Notes for Lectures 16 and 17

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Reflected random walk & positive recurrence. The main topic of these two lectures is the reflected random walk (RRW) example from Sect. 1.11 of the text. This is motivation for the concept of positive recurrence, which is central to understanding the existence stationary distributions for Markov chains with infinite state spaces.

The main result about the RRW is the following:

- If $0 < p < \frac{1}{2}$, then there exists a (unique) stationary distribution π such that $\pi(x) > 0$ for all x. The chain is recurrent, and $E_x T_x < \infty$ for all x.
- If $\frac{1}{2} < p < 1$, then there is no stationary distribution, and all states are transient.
- If $p = \frac{1}{2}$, then there is no stationary distribution, and all states are recurrent but $E_x T_x = \infty$.

I mainly followed the text's proof. Here are some comments:

1) A special property of the RRW is that *if* there is a stationary distribution, then it satisfies detailed balance. This is not hard to show; you are asked to do this on Homework 8. This property allows one to easily find an explicit expression for the stationary distribution (if it exists).

2) The text treats the three cases above separately. But one can unify the analysis. In particular, if we fix $N > 0$ and let $h_N(x) = P_x(V_N < V_0)$, then using the method of Sect. 1.9 (exit distributions) we find (see Example 1.44 from Sect. 1.9)

$$h_N(x) = \begin{cases} \frac{1-(q/p)^x}{1-(q/p)^N}, & p \neq \frac{1}{2} \\ \frac{x}{N}, & p = \frac{1}{2} \end{cases} \tag{1}$$

Letting $N \to \infty$ and assuming $V_N \to \infty$, we get

$$P_x(V_0 = \infty) = \begin{cases} 1 - (q/p)^x, & p > \frac{1}{2} \\ 0, & p \leqslant \frac{1}{2} \end{cases} \tag{2}$$

so

$$P_x(V_0 < \infty) = \begin{cases} (q/p)^x, & p > \frac{1}{2} \\ 1, & p \leqslant \frac{1}{2} \end{cases} \tag{3}$$

Thus, for $x > 0$, we find that $\rho_{x0} = P_x(T_0 < \infty) = 1$ when $p \leqslant \frac{1}{2}$, but $\rho_{x0} = P_x(T_0 < \infty) < 1$ when $p > \frac{1}{2}$. Using the kind of arguments we learned in Sect. 1.3, one can then prove that 0 is recurrent when $p \leqslant \frac{1}{2}$ and transient when $p > \frac{1}{2}$.

3) To show that $E_0 T_0 = \infty$ when $p = \frac{1}{2}$, we use

$$E_0 T_0 = \frac{1}{2} \cdot 1 + \frac{1}{2} E_1 V_0. \tag{4}$$

The quantity $E_1 V_0$ can be calculated using the methods of Sect. 1.10 (exit times) as follows: let

$$V_{0,N} = \min\{n \geqslant 0 \mid X_n = 0 \text{ or } X_n = N\}. \tag{5}$$

Let $g_N(x) = E_x V_{0,N}$. Then

$$g_N(x) = \frac{1}{2} g_N(x-1) + \frac{1}{2} g_N(x+1) + 1 \tag{6}$$

and $g_N(0) = g_N(N) = 0$. One can check (homework!) that

$$g_N(x) = x(N-x) \tag{7}$$

is the solution. In particular, $g_N(1) = N - 1 = E_1 V_{0,N}$. As $N \to \infty$, we have $E_1 V_{0,N} \to \infty$. But we expect $V_{0,N} \to V_0$. So $E_1 V_0 = \infty$.