# Notes for Lecture 18

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**Theorem 1.30.** Today we proved Theorem 1.30, which states that an irreducible chain has a stationary distribution if and only if there is at least one positive recurrent state (and if one state is positive recurrent, all states are positive recurrent). I just followed the proof in the text, which leans on Theorem 1.24 on the existence of stationary distributions.

**Branching process.** For the analysis of branching processes, I followed the book closely. Let $\varepsilon_n = P_1(X_n = 0)$ be the probability of extinction at or before time $n$. (The book called this $\rho_n$, which I don't like because $\rho$ looks like $p$.) Then

$$\varepsilon_n = \varphi(\varepsilon_{n-1}) \tag{1}$$

where

$$\varphi(\theta) = \sum_{k=0}^{\infty} p_k \theta^k. \tag{2}$$

As mentioned in lecture, $\varphi$ is very much like the moment generating function (MGF) for the number of offsprings $Y$. Indeed, the MGF of $Y$ is defined as

$$M_Y(s) \ = \ E(e^{sY}) \ = \ \sum_{k=0}^{k} p_k e^{sk}. \tag{3}$$

So the relationship between the two is $M_Y(s) = \varphi(e^s)$.

You can easily verify that

$$\varphi(1) = 1 \tag{4a}$$

$$\varphi'(1) = \mu \tag{4b}$$

$$\varphi'(\theta), \varphi''(\theta) \geqslant 0 \text{ for } \theta \geqslant 0 \tag{4c}$$

The last line says $\varphi$ is increasing and concave up (or at least not concave down).

I prefer to make the rest of the argument by drawing pictures. Here is a short video (and a copy of the "board") explaining this via "cobweb" diagrams. The bottom line is that assuming $p_0 > 0$,

- $\mu \leqslant 1$ implies that $P_1(T_0 < \infty) = 1$. For "version B" of the model, where we set $p(0,1) = 1$, this implies $x = 0$ is recurrent.
- $\mu > 1$ implies that $p_0 < P_1(T_0 < \infty) < 1$, and the probability $\varepsilon_\infty = P_1(T_0 < \infty)$ is the smallest solution of the equation $\varepsilon_\infty = \varphi(\varepsilon_\infty)$. For "version B," this implies $x = 0$ is transient.