Notes for Lecture 26

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April 23, 2020

Today, I covered

1) Theorems 2.15, which essentially states that conditioned on $N(t) = n$, the set of arrival times $\{T_1, \cdots, T_n\}$ have the same distribution as $\{U_1, \cdots, U_n\}$ where the U_i are IID uniform random variables on $[0, t]$.

2) Theorem 2.16, which states that conditioned on $N(t) = n$, the number of events occurring in an interval $I \subset [0, t]$ is binomial, with $p = \text{length}(I)/t$.

3) Discussed the proof of Theorem 2.15. Some notes:

   - Theorem 2.16 is an easy consequence of 2.15.
   - Theorem 2.14, on superpositions, is easy to prove. It just follows from the fact that a sum of independent Poisson random variables is another Poisson random variable.
   - I had planned to talk the proof of Theorem 2.11, on thinning Poisson processes, but ran out of time. See the explanation in the text instead.

For the proof of Theorem 2.15, I largely followed the text, but expanded some details. Here it is. For concreteness, let $n = 2$. Our goal is to show that (T_1, T_2) has density

$$f(t_1, t_2) = \begin{cases} 2!/t^2, & 0 \leqslant t_1 \leqslant t_2 \leqslant t \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

Note that this is the density you would expect for two random variables $V_1 < V_2$, obtained from two independent uniform random variables (U_1, U_2) by sorting, i.e., $V_1 = U_1 \wedge U_2$ and $V_2 = U_1 \vee U_2$.

The first step is to recall that for two continuous random variables X and Y, we have

$$P((X, Y) \in E) \approx f_{XY}(x_0, y_0) \cdot \text{area}(E) \tag{2}$$

if their joint PDF f_{XY} is sufficiently smooth, the set E is sufficiently small (in diameter, say), and (x_0, y_0) is any point in E. Moreover, the quality of the approximation improves as the diameter of E goes to 0. This is just because that if the joint PDF is smooth, then on a small set it will be very close to being constant.

Next, let $0 < t_1 < t_2 < t$, and fix a small $\delta > 0$. Observe

$$P(T_1 \in [t_1, t_1 + \delta], T_2 \in [t_2, t_2 + \delta] \mid N(t) = 2) \tag{3a}$$

$$= \frac{P(T_1 \in [t_1, t_1 + \delta], T_2 \in [t_2, t_2 + \delta], N(t) = 2)}{P(N(t) = 2)} \tag{3b}$$

by definition of conditional probability, which can be rewritten as

$$= \frac{P(T_1 \in [t_1, t_1 + \delta], T_2 \in [t_2, t_2 + \delta], T_3 > t)}{P(N(t) = 2)} \tag{3c}$$

$$= \frac{P(\tau_1 \in [t_1, t_1 + \delta], \tau_1 + \tau_2 \in [t_2, t_2 + \delta], \tau_3 > t - t_2)}{P(N(t) = 2)}. \tag{3d}$$

where we have introduced the waiting times $\tau_1 = T_1$, $\tau_2 = T_2 - T_1$, and $\tau_3 = T_3 - T_2$. Doing this is useful because as we know, the waiting times are indepenedent exponential random variables.

We now apply Eq. (1), so that

$$\frac{P(\tau_1 \in [t_1, t_1 + \delta], \tau_1 + \tau_2 \in [t_2, t_2 + \delta], \tau_3 > t - t_2)}{P(N(t) = 2)} \tag{4a}$$

$$= \frac{P((\tau_1, \tau_2) \in A, \tau_3 > t - t_2)}{P(N(t) = 2)} \tag{4b}$$

$$\approx \frac{\lambda e^{-\lambda t_1} \cdot \lambda e^{-\lambda(t_2 - t_1)} \cdot e^{-\lambda(t - t_2)} \cdot \delta^2}{e^{-\lambda t}(\lambda t)^2/2!}. \tag{4c}$$

In the above, the set A is a parallelogram with base and height δ (and thus area δ^2). This is a geometric consequence of rewriting the probability in terms of (τ_1, τ_2) instead of (T_1, T_2). To see this, observe that the event $(\tau_1, \tau_1 + \tau_2) \in [t_1, t_1 + \delta] \times [t_2, t_2 + \delta]$ corresponds to a square in the τ_1-τ_2 plane. This event is equivalent to the event $(\tau_1, \tau_2) \in A$ for some subset A of the plane. To see that A has the geometry claimed above, observe that

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \tau_1 \\ \tau_1 + \tau_2 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}. \tag{5}$$

Thus A is the image of the square $[t_1, t_1 + \delta] \times [t_2, t_2 + \delta]$ under the linear transformation defined by the 2×2 matrix above. Since the matrix has determinant 1, the area of the image is the same as that of the square, namely δ^2. Finally, comparing Eqs. (4c) and (1), we see that the conditional PDF of (T_1, T_2) is Eq. (4c) without the δ^2 factor.

The rest is algebra: Eq. (4c), after simplification, is just $2/t^2$, which is what we wanted. This argument generalizes to show that conditioned on $N(t) = n$, the arrival times $(T_1, \cdots, T_n)$ have joint pdf

$$f(t_1, \cdots, t_n) = \begin{cases} n!/t^n, & 0 < t_1 < \cdots < t_n < 1 \\ 0, & \text{otherwise} \end{cases} \tag{6}$$