Diffusion for a Markov, Divergence-form Generator

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Abstract

We consider the long-time evolution of solutions to a Schrödinger-type wave equation on a lattice with a Markov random generator. We show that solutions to this problem possess a diffusive scaling limit and compute higher moments.

Based on joint work with Jeffrey Schenker.

Statement of the Theorem

Theorem

If $\psi_t \in \ell^2(\mathbb{Z}^d)$ *satisfies*

$$\begin{cases} i\partial_t \psi_t(x) & = & \nabla^\dagger \theta_{\omega(t)} \nabla \psi_t(x) \\ \psi_0(x) & = & \delta_0(x) \end{cases},$$

then

$$\lim_{\eta \to 0^+} \sum_{x \in \mathbb{Z}^d} e^{i\sqrt{\eta} k \cdot x} \mathbb{E}\left(|\psi_{t/\eta}(x)|^2 \right) = e^{-4t \sum_{e_1, e_2} (k \cdot e_1)(k \cdot e_2) D_{e_1, e_2}}.$$

What do we mean by diffusion?

- Consider the standard heat equation

$$\begin{cases} \partial_t u(x,t) = \Delta u(x,t) & (x,t) \in \mathbb{R}^d \times \mathbb{R}^+ \\ u(x,0) = \delta_0(x) & x \in \mathbb{R}^d \end{cases}$$

with solution $u(x,t) = (2\pi t)^{-d/2} e^{-|x|^2/4t}$.

- $x \mapsto c_t u(x,t)$ is a p.d.f. on $\mathbb{R}^d$ with $c_t = \left(\int_{\mathbb{R}^d} u(x,t)\, dt \right)^{-1}$ the normalizing constant.

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- The pth moment of position is given by

$$\int_{\mathbb{R}^d} |x|^p \, c_t u(x,t)\, dx = \frac{c_t \omega_d}{(2\pi t)^{d/2}} \int_0^\infty r^{p+d-1} e^{-\frac{r^2}{4t}}\, dr,$$

  where $\omega_d = |\partial \mathcal{B}(0,1)|$ is the surface area of the unit ball in $\mathbb{R}^d$.
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- **Definition: Diffusive Scaling**

$$\begin{cases} t \mapsto \frac{1}{\eta} t \\ x \mapsto \frac{1}{\sqrt{\eta}} x \end{cases} \quad \text{as } \eta \to 0^+$$

- Question: The problem under consideration is defined on the lattice $\mathbb{Z}^d$. How do we scale a discrete space?
- Answer: Mollify.

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Characterization of Diffusion for a Discrete Problem

- $h \in C_c^\infty(\mathbb{R}^d)$, $\int h\,dx = 1$, $h \geq 0$
- Under diffusive scaling, if the convolution $h * |\psi_t|^2$ converges (weakly) to a solution of the heat equation, then we say that the model exhibits *diffusion*.
- A Fourier transform removes the mollifier from our diffusion criterion.
- **Diffusion Criterion:**

$$\sum_{x \in \mathbb{Z}^d} e^{i\sqrt{\eta}k\cdot x} |\psi_{t/\eta}(x)|^2 \to e^{-Dt|k|^2}, \quad k \in \mathbb{T}^d$$

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Resolvent Analysis

- Key step:

$$\sum_{x \in \mathbb{Z}^d} e^{i\sqrt{\eta}k\cdot x}\mathbb{E}(|\psi_{t/\eta}(x)|^2)$$

$$= -\frac{1}{2\pi i}\int_\Gamma e^{-tz}\left\langle \delta_0 \otimes 1, \frac{\eta}{i\hat{L}_{\sqrt{\eta}k} + B - \eta z}\delta_0 \otimes 1 \right\rangle dz$$

- Notes:
  - The LHS is (almost) the diffusion criterion.
  - The expectation allows us to use a Feynman-Kac-Pillet formula.
  - FKP allows us to express the expectation as a matrix element of the semigroup $e^{-t(i\hat{L}_{\sqrt{\eta}k}+B)}$,
  - which can be understood by the holomorphic functional calculus:

$$e^{t(i\hat{L}_{\sqrt{\eta}k}+B)} = \frac{1}{2\pi i}\int_\Gamma e^{tz}\frac{1}{i\hat{L}_{\sqrt{\eta}k} + B - z}dz$$

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- We have reduced the problem to understanding:

$$\lim_{\eta \to 0^+} \left\langle \delta_0 \otimes 1, \frac{\eta}{i\hat{L}_{\sqrt{\eta}k} + B - \eta z} \delta_0 \otimes 1 \right\rangle .$$

- From here,
  - use projections and the Schur complement formula.
  - construct a symmetric operator D_k, which is a lower bound for the matrix element in question. Use this to show the limit exists and is of the desired form.
- Higher Moments?

$$\lim_{\eta \to 0^+} \sum_{x \in \mathbb{Z}^d} e^{i\sqrt{\eta}k \cdot x} \mathbb{E}\left(|\psi_{t/\eta}(x)|^2 \right) = e^{-4t \sum_{e_1, e_2} (k \cdot e_1)(k \cdot e_2) D_{e_1, e_2}},$$

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