Spectral Theory of a Canonical System

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Introduction of a canonical system

A canonical system is a family of differential equations of the form

$$Ju'(x) = zH(x)u(x), \quad z \in \mathbb{C}. \tag{1}$$

$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $H(x)$ is a 2×2 positive semidefinite matrix.
Assume H does not vanish on any open interval.

Consider the Hilbert space

$$L^2(H, \mathbb{R}_+) = \left\{ f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} : \|f\| < \infty \right\}$$

with inner product $\langle f, g \rangle = \int_0^\infty f(x)^* H(x) g(x) dx$

Symmetric relation on any Hilbert space $\mathcal{H}$

A linear relation $\mathcal{R} = \{(f,g) : f, g \in \mathcal{H}\}$ on $\mathcal{H}$ is a subspace of $\mathcal{H}^2$.

Domain: $D(\mathcal{R}) = \{f \in \mathcal{H} : \quad (f,g) \in \mathcal{R}\}$

Range: $R(\mathcal{R}) = \{g \in \mathcal{H} : \quad (f,g) \in \mathcal{R}\}$

Inverse: $\mathcal{R}^{-1} = \{(g,f) : (f,g) \in \mathcal{R}\}$

Adjoint of $\mathcal{R}$ in $\mathcal{H}^2$:

$$\mathcal{R}^* = \{(h,k) \in \mathcal{H}^2 : \ \langle f,k \rangle = \langle g,h \rangle \text{ for all } (f,g) \in \mathcal{R}\}.$$

A linear relation $\mathcal{S}$ is called symmetric if $\mathcal{S} \subset \mathcal{S}^*$ and self-adjoint if $\mathcal{S} = \mathcal{S}^*$.
Goal: Discuss the spectrum of such self adjoint relation. Let $(z - \mathcal{R}) = \{(f, zf - g) : \ (f, g) \in \mathcal{R}\}$, $\mathcal{R}_z = R(z - \mathcal{R})$ and $N(\mathcal{R}, z) = \{f : (f, zf) \in \mathcal{R}\}$. Observe that,

$$N(\mathcal{R}^*, \bar{z}) = \mathcal{R}_z^{\perp}.$$

The regularity domain of $\mathcal{R}$ is the set

$$\Gamma(\mathcal{R}) = \Big\{ z \in \mathbb{C} : \exists C(z) > 0 : \|(zf - g)\| \geq C(z)\|f\|, \ \forall (f, g) \in \mathcal{R} \Big\}.$$

$\Gamma(\mathcal{R})$ satisfies the following properties:

1. $z \in \Gamma(\mathcal{R})$ if and only if $(z - \mathcal{R})^{-1}$ is a bounded linear operator on $D(\mathcal{R})$.
2. If $\mathcal{R}$ is symmetric, then $\mathbb{C} - \mathbb{R} \subset \Gamma(\mathcal{R})$.
3. $\Gamma(\mathcal{R})$ is open.

$\beta(\mathcal{R}, z) = \dim \mathcal{R}_z^{\perp}$ is called the defect index of $\mathcal{R}$ and z.

Theorem 1

The defect index $\beta(\mathcal{R}, z)$ is constant on each connected subset of $\Gamma(\mathcal{R})$. If $\mathcal{R}$ is symmetric, then the defect index is constant in the upper and lower half-planes.

For $z \in \mathbb{C}^+, m = \beta(\mathcal{R}, z)$ and for $w \in \mathbb{C}^-, n = \beta(\mathcal{R}, w)$ are written as a pair (m, n), called the defect indices of $\mathcal{R}$.

Theorem 2

Let $\mathcal{R}$ be a closed symmetric relation on a Hilbert space $\mathcal{H}$ with defect indices (m, n) then

1. *$\mathcal{R}$ possess self-adjoint extension if and only if its defect indices are equal$(m = n)$.*
2. *A symmetric extension $\mathcal{R}'$ of $\mathcal{R}$ is self-adjoint if and only if $\mathcal{R}'$ is an $m-$dimensional extension of $\mathcal{R}$.*

The resolvent set for a closed relation $\mathcal{R}$ is a set

$$\rho(\mathcal{R}) = \Big\{ z \in \mathbb{C} : \exists\, T \in B(\mathcal{H}) : \mathcal{R} = \{(Tf, zTf - f) : \; f \in \mathcal{H}\} \Big\}$$

and the spectrum of $\mathcal{R}$ is

$$\sigma(\mathcal{R}) = \mathbb{C} - \rho(\mathcal{R})$$

We call $S(\mathcal{R}) = \mathbb{C} - \Gamma(\mathcal{R})$ the spectral kernel of $\mathcal{R}$.

Theorem 3

Let $\mathcal{T}$ is a self-adjoint relation on $\mathcal{H}$. Suppose $z \in \Gamma(\mathcal{T})$ and $T = (\mathcal{T} - z)^{-1}$ then

1. $S(\mathcal{T}) = \sigma(\mathcal{T})$
2. *If* $\lambda \in \Gamma(T)$ *then* $(z - \frac{1}{\lambda}) \in \Gamma(\mathcal{T})$.
3. *If* $\lambda \in S(\mathcal{T})$ *then* $\frac{1}{z-\lambda} \in S(T)$.
4. $S(T) \subset \sigma(T)$.

Relation induced by a Canonical System on $L^2(H,\mathbb{R}_+)$

Consider the maximal relation $\mathcal{R}$ on $L^2(H,\mathbb{R}_+)$ given by

$$\mathcal{R} = \{(f,g) \in \left(L^2(H,\mathbb{R}_+)\right)^2 : \ f \in AC, Jf' = Hg\}.$$

The adjoint relation $\mathcal{R}_0 = R^*$, called as minimal relation is defined by

$$\mathcal{R}_0 = \{(f,g) \in \left(L^2(H,\mathbb{R}_+)\right)^2 : \langle g,h\rangle = \langle f,k\rangle \ \text{ for all } (h,k) \in \mathcal{R}\}$$

The minimal relation $\mathcal{R}_0$ is symmetric: $\mathcal{R}_0 \subset \mathcal{R}_0^* = \mathcal{R}$ and is given by

$$\mathcal{R}_0 = \{(f,g) \in \mathcal{R} : f(0+) = 0, \lim_{x\to\infty} f^*(x)Jh(x) = 0, (h,k) \in \mathcal{R}\}.$$

- $\beta(\mathcal{R}_0)$ is equal to the number of linearly independent solutions of the system 1 of whose class lie in $L^2(H, \mathbb{R}_+)$.

It follows that $\mathcal{R}_0$ has equal defect indices, by Theorem 2 it has a self-adjoint extension say $\mathcal{T}$.
Note: the limit circle case of the system 1. That implies for any $z \in \mathbb{C}^+$ the deficiency indices of $\mathcal{R}_0$ are $(2, 2)$. Suppose $p \in D(\mathcal{R}) \setminus D(\mathcal{R}_0)$ such that $\lim_{x\to\infty} p(x)^* Jp(x) = 0$. Then the relation

$$\mathcal{T}^{\alpha,p} = \{(f, g) \in \mathcal{R} : f_1(0)\sin\alpha + f_2(0, z)\cos\alpha = 0 \text{ and } \lim_{x\to\infty} f(x)^* Jp(x) = 0\}.$$

defines a self-adjoint relation.

We next discuss the spectrum of $\mathcal{T}^{\alpha,p}$. Let $u(x,z)$ and $v(x,z)$ be two linearly independent solutions of the system 1 with

$$u(0,z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v(0,z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Let $z \in \mathbb{C}^+$ and write $f(x,z) = u(x,z) + m(z)v(x,z) \in L^2(H, \mathbb{R}_+)$ satisfying $\lim_{x\to\infty} f(x,z)^* JP(x) = 0$. Let $T(x,z) = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}$ and $w_\alpha(x,z) = \frac{1}{\sin\alpha + m(z)\cos\alpha} T(x,z) \begin{pmatrix} \cos\alpha \\ -\sin\alpha \end{pmatrix}$.

Let $z \in \rho(\mathcal{T}^{\alpha,p})$ then the resolvent operator $(\mathcal{T}^{\alpha,p} - z)^{-1}$ is given by

$$(\mathcal{T}^{\alpha,p} - z)^{-1} h(x) = \int_0^\infty G(x,t,z) H(t) h(t) dt$$

where $G(x,t,z) = \begin{cases} f(x,z) w_\alpha(t,\bar{z})^* & \text{if } 0 < t \le x \\ w_\alpha(t,\bar{z}) f(x,\bar{z}) & \text{if } x < t \le \infty \end{cases}$
This is unitarily equivalent with the integral operator (Hilbert Schmidt) $\mathcal{L}$ on $L^2(I, \mathbb{R}_+)$ given by

$$(\mathcal{L}g)(x) = \int_0^\infty L(x,t) g(t) dt, \;\; L(x,t) = H^{\frac{1}{2}}(x) G(x,t,z) H^{\frac{1}{2}}(t).$$

Hence it has only discrete spectrum consisting of eigenvalues and possibly zero. By Theorem 3, $\mathcal{T}^{\alpha,p}$ has discrete spectrum consisting of eigenvalues.

Theorem 4

The defect index $\beta(\mathcal{R}_0, z) = \textit{ dim } \mathcal{R}_{0_z}^{\perp} = \textit{ dim } N(\mathcal{R}, \bar{z})$ *of* $\mathcal{R}_0$ *is constant on* $\mathbb{C}$*.*

Proof.

Since $\mathcal{R}_0$ is a symmetric relation, by Theorem 1 the defect index $\beta(\mathcal{R}_0, z)$ is constant on upper and lower half planes. Suppose $\beta(\mathcal{R}_0, \lambda) < 2$ for some $\lambda \in \mathbb{R}$. Since $\Gamma(\mathcal{R}_0)$ is open, $\lambda \notin \Gamma(\mathcal{R}_0)$ and hence $\lambda \in S(\mathcal{R}_0)$. Since for each $\alpha \in (0, \pi]$, $\mathcal{T}^{\alpha,p}$ is self-adjoint extension of $\mathcal{R}_0$, $\lambda \in S(\mathcal{T}^{\alpha,p}) = \sigma(\mathcal{T}^{\alpha,p})$. Since $\sigma(\mathcal{T}^{\alpha,p})$ consists of only eigenvalues, λ is an eigenvalue for all boundary conditions α at 0. However, this is impossible unless $\beta(\mathcal{R}_0, \lambda) = 2$. This completes the proof. □

Theorem 5

Consider the canonical system 1 with trace$H \equiv 1$ then it prevails limit point case.

Proof.

Suppose it prevails the limit circle case. That means all solutions of 1 are in $L^2(H, \mathbb{R}_+)$. By Theorem 4, for $0 \in \mathbb{R}$, $\dim N(\mathcal{R}, 0) = 2$. In particular, 0 is an eigenvalue and $u(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are the eigenfunctions of the relation $\mathcal{R}$ in $L^2(H, \mathbb{R}_+)$. However, $\int_0^\infty u(x)^* H(x) u(x) dx + \int_0^\infty v(x)^* H(x) v(x) dx = \int_0^\infty \text{trace} H(x) dx = \infty$. This is a contradiction. It follows that the canonical system 1 has limit point case. □

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Thank You!