Bose-Einstein condensation on fractal spaces

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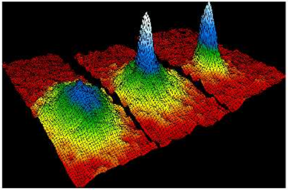
Bose gas inside a GSC

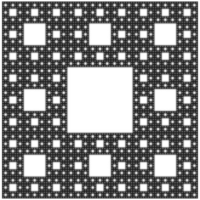


Goal: Study equilibrium thermodynamics of quantum gases in fractal spaces.
We will focus on a gas of quantum particles which satisfy Bose-Einstein statistics: the many-body wavefunction is unchanged under particle swapping.

Questions

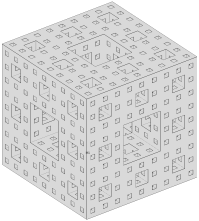
- How does the fractal geometry (with non-integer Hausdorff dimension) affect the thermodynamics (energy, pressure, ...) of the confined quantum gas?
- Or, is the Hausdorff dimension the *right* dimension to consider in this context?

Generalized Sierpinski carpets



Sierpinski carpet

($m_F = 8$, $l_F = 3$)



Menger sponge

($m_F = 20$, $l_F = 3$)

- GSCs are highly symmetric, infinitely ramified fractals and have connected interior. (*Translation:* Analysis is much harder than, say, Sierpinski gaskets.)
- Let l_F and m_F be, resp. the length and mass scale factor of a GSC F. Hausdorff dimension $d_h(F) = \log m_F / \log l_F$.
- F_n : nth-level approximation of the fractal F.
- Let ν_n be the self-similar measure on F_n (= Borel prob measure which assigns equal weight to every cell of F_n). $\nu_n \rightharpoonup \nu =$ (const.)$\times d_h(F)$-dim Hausdorff measure.

Gibbs state & partition function [in the grand canonical ensemble]

- $\mathcal{H}_n = \mathrm{Sym}\left(L^2(F,\nu)^{\otimes n}\right)$, $\mathcal{F} := \oplus_{n\geq 0}\mathcal{H}_n$ (bosonic Fock space).
- $H : \mathcal{H}_1 \to \mathcal{H}_1$ a one-body Hamiltonian, $\mathbf{H} = d\Gamma(H)$ its second quantization, $\mathbf{N} = d\Gamma(1)$ the number operator.
- $\beta \in (0,\infty]$ is the inverse temperature.
- $\mu \in (-\infty, \inf \mathrm{Spec}(H)]$ is the chemical potential.

The Gibbs state is a linear functional over the quasi-local C^*-algebras on $\mathcal{F}$ satisfying

$$\omega_{\beta,\mu}(\mathbf{A}) = \Xi_{\beta,\mu}^{-1}\mathrm{Tr}_{\mathcal{F}}\left(\mathbf{A}e^{-\beta(\mathbf{H}-\mu\mathbf{N})}\right) \quad \forall \mathbf{A} \in C^*\text{-algebras}$$

where the partition function reads

$$\Xi_{\beta,\mu} := \mathrm{Tr}_{\mathcal{F}}\left(e^{-\beta(\mathbf{H}-\mu\mathbf{N})}\right).$$

The free energy is

$$F_{\beta,\mu} = -\beta^{-1}\log\Xi_{\beta,\mu}.$$

The expected values of particle number and energy can be obtained by taking derivatives:

$$\omega_{\beta,\mu}(\mathbf{N}) = \frac{1}{\beta}\frac{\partial}{\partial\mu}\log\Xi = \mathrm{Tr}_{\mathcal{H}_1}\frac{1}{e^{\beta(H-\mu)}-1}\ , \quad \omega_{\beta,\mu}(\mathbf{H}) = -\frac{\partial}{\partial\beta}\log\Xi.$$

Special case: Bose gas, no inter-particle interactions.

$$\log\Xi_{\beta,\mu} = -\mathrm{Tr}_{\mathcal{H}_1}\log(1-e^{-\beta(H-\mu)}).$$

Statement of the Problem

$$\log \Xi_{\beta,\mu} = -\mathrm{Tr}_{\mathcal{H}_1} \log(1 - e^{-\beta(H-\mu)}), \quad \omega_{\beta,\mu}(\mathbf{N}) = \mathrm{Tr}_{\mathcal{H}_1} \frac{1}{e^{\beta(H-\mu)} - 1}.$$

Compute $\log \Xi_{\beta,\mu}$ and/or $\omega_{\beta,\mu}(\mathbf{N})$ when F is a GSC, ν is the self-similar measure on F, and the Bose gas is of the following types:

1. Atomic gas: $H = -\Delta$, $\mu \leq \inf \mathrm{Spec}(H)$. [Satisfies Schr eqn $i\partial_t \psi = -\Delta\psi$.]
   $\rightarrow$ Bose-Einstein condensation
2. (Massless) photon gas: $H = \sqrt{-\Delta}$, $\mu = 0$. [Satisfies wave eqn $\partial_{tt}\psi = \Delta\psi$.]
   $\rightarrow$ Blackbody radiation, Casimir effect

Here the Laplacian Δ is defined à la Barlow-Bass (BMs on outer approximations) or Kusuoka-Zhou (random walks on SC graphs). Up to time change, both versions of the Laplacian generate the same (and the unique) Brownian motion on SC [Barlow-Bass-Kumagai-Teplyaev '10].

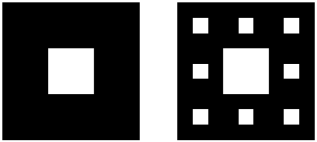
To demonstrate BEC, we need to show that in the infinite volume limit, $\exists k \in \mathbb{N}_0$ such that the particle density projected onto the $(k+1)$-th eigenfunction is strictly positive

$$\liminf_{\mathrm{Vol}\to\infty} \omega_{\beta,\mu}\left(\frac{\mathbf{P}_k \mathbf{N} \mathbf{P}_k}{\mathrm{Vol}}\right) = \liminf_{\mathrm{Vol}\to\infty} \frac{1}{\mathrm{Vol}} \frac{1}{e^{\beta(E_k-\mu)} - 1} > 0$$

for sufficiently large β.

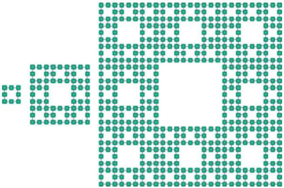
Laplacian (equiv. B.M., Dirichlet form) on GSCs

Outer Approximation



W^n_t: reflecting B.M. on F_n.
Barlow-Bass ('90s) showed that there exist $a_n \asymp (m_F \rho_F / l_F^2)^n$ such that $\{X^n_t\}_n := \{W^n_{a_n t}\}_n$ converges in subsequence.

Inner approximation



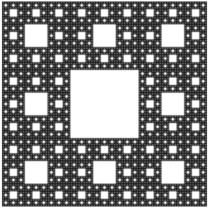
Take the graph energy on $G_n = (V_n, E_n)$,
$\mathcal{E}_n(u) = \sum_{\langle xy \rangle \in E_n} [u(x) - u(y)]^2$.
Kusuoka-Zhou ('92) showed that $\{\rho^n_F \mathcal{E}_n\}_n$ converges in subsequence.

The Laplacian is defined to be the infinitesimal generator of the limit process on the GSC F.

Theorem (Barlow-Bass-Kumagai-Teplyaev '10)

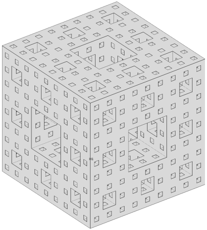
Up to deterministic time change, both versions of the Laplacian generates the unique B.M. on F which respects the local symmetries of F.

Various dimensions of the Sierpinski carpet



Sierpinski carpet

$(m_F = 8, l_F = 3)$



Menger sponge

$(m_F = 20, l_F = 3)$

- Hausdorff dim: $d_h = \log m_F / \log l_F$.
- Walk dim: $\mathbb{E}^x \tau_{B_r(x)} \asymp r^{d_w}$. [On fractals, $d_w > 2$ (sub-Gaussian diffusion).]
- Spectral dim: $d_s = 2(d_h/d_w) = 2 \log m_F / \log(m_F \rho_F)$.
- Resistance renormalization factor ρ_F: Relates the resistance on F_n to that on F_{n+1}. No closed form expression of ρ_F is known on GSCs [Barlow-Bass '99] .

$$\rho_F < 1 \Leftrightarrow d_s(F) > 2 \Leftrightarrow \text{BM is transient on the unbounded carpet.}$$

Weyl asymptotics of the Laplacian on GSC

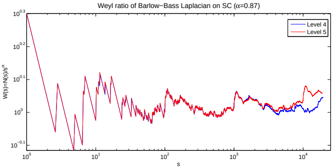
Let $N(s) := \#\{\lambda < s : \lambda \text{ an eigenvalue of } -\Delta\}$ be the integrated DOS.

Heuristic (as supported by lots of numerics: C.-Strichartz, Begué-Kalloniatis-Strichartz)

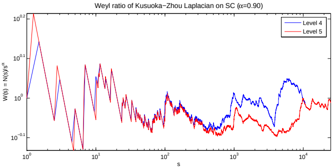
$$N(s) = s^{d_h/d_w}\left[H(\log s) + o(1)\right] \quad \text{as} \quad s \to \infty,$$

where H is $\log(m_F \rho_F)$-periodic, and bounded away from 0 and ∞.

The log-periodic modulation is due to the discrete scale invariance of the carpet.



(c) Outer approx



(d) Inner approx

Problem: We don't know how to prove this heuristic (involves sophisticated Tauberian arguments). So to show BEC rigorously we use a different (albeit general & powerful) technique: heat kernel estimates & spectral zeta functions.

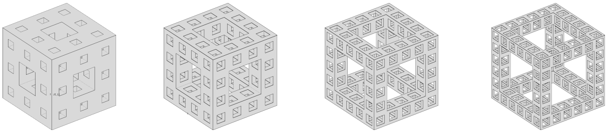
Criterion for BEC in Sierpinski carpets

Theorem (C.)

For an unbounded GSC, the following are equivalent:

1. Spectral dimension $d_s > 2$.
2. (The Brownian motion whose generator is) the Laplacian is transient.
3. BEC exists for a low-temperature, high-density ideal Bose gas.

BEC in non-integer dimensions: Menger Sponges



	MS(3,1)	MS(4,2)	MS(5,3)	MS(6,4)
d_h	$\log_3 20 \approx 2.73$	$\log_4 32 = 2.5$	$\log_5 44 \approx 2.35$	$\log_6 56 \approx 2.25$
Rigorous bnds on d_s [Barlow-Bass '99]	$2.21 \sim 2.60$	$2.00 \sim 2.26$	$1.89 \sim 2.07$	$1.82 \sim 1.95$
Numerical d_s [C.-Strichartz]	2.51...	-	2.01...	-
BEC exists?	Yes	Yes	Yes (?)	No

Heat kernel estimates on GSCs

Heat kernel: $\mathbb{E}^x[f(X_t)] = (e^{t\Delta}f)(x) = \int_F p_t(x,y) f(y)\nu(dy)$.

Theorem (Barlow, Bass, Kusuoka, Zhou, $\cdots$)

$$p_t(x,y) \asymp C_1 t^{-d_h/d_w} \exp\left(-C_2 \left(\frac{|x-y|^{d_w}}{t}\right)^{\frac{1}{d_w-1}}\right).$$

Here $d_h = \log m_F / \log l_F$ (Hausdorff), $d_w = \log(\rho_F m_F)/\log l_F$ (walk).

$$d_s = 2\frac{d_h}{d_w} = 2\frac{\log m_F}{\log(m_F \rho_F)}$$ is the spectral dimension of the carpet.

For any carpet, $d_w > 2$ and $1 \leq d_s < d_h$, indicative of sub-Gaussian diffusion.

Theorem (Hambly '11, Kajino '10)

There exists a $\log(m_F \rho_F)$-periodic function G, bounded away from 0 and ∞, such that the heat kernel trace

$$K(t) := \mathrm{Tr}(e^{t\Delta}) = \int_F p_t(x,x)\nu(dx) = t^{-d_h/d_w}\left[G(-\log t) + o(1)\right] \quad \text{as } t \downarrow 0.$$

Short-time asymptotics of the heat kernel trace on GSC

Theorem (Kajino '09+)

For any GSC $F \subset \mathbb{R}^d$ with Dirichlet exterior boundary, there exist continuous, $\log(m_F \rho_F)$-periodic functions $G_k : \mathbb{R} \to \mathbb{R}$ for $k = 0, 1, \cdots, d$ such that

$$K(t) = \sum_{k=0}^{d} t^{-d_k/d_w} G_k(-\log t) + \mathcal{O}\left(\exp\left(-ct^{-\frac{1}{d_w - 1}}\right)\right) \quad \text{as} \quad t \downarrow 0,$$

where $d_k := d_h(F \cap \{x_1 = \cdots = x_k = 0\})$.

Proof exploits the full symmetry of the GSC and uses the renewal thm.
Rem 1. $G_0 > 0$ and $G_1 < 0$. Numerics suggest that G_0 is nonconstant.
Rem 2. Corresponding Weyl asymptotics for a parallelpiped in $\mathbb{R}^d$ is

$$K(t) = \sum_{k=0}^{d} t^{-(d-k)/2} C_k + \mathcal{O}(\exp(-ct^{-1})) \quad \text{as} \quad t \downarrow 0.$$

Notation. $G_k(x) = \sum_{p \in \mathbb{Z}} \hat{G}_{k,p} \exp\left(\frac{2p\pi i x}{\log(m_F \rho_F)}\right)$.

Definition (Cesaro-averaged spectral volume)

The spectral volume of a GSC of side L is $V_{\mathrm{S}} = (4\pi)^{d_s} \hat{G}_{0,0} L^{d_s}$. ($d_s = 2(d_h/d_w)$)

Spectral zeta function

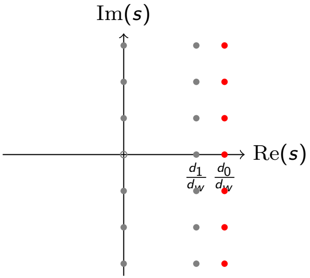
$$\zeta_\Delta(s,\gamma) = \mathrm{Tr}\,\frac{1}{(-\Delta+\gamma)^s} = \frac{1}{\Gamma(s)}\int_0^\infty t^s K(t) e^{-\gamma t}\,\frac{dt}{t}$$

Absicssa of convergence is $\mathrm{Re}(s) = d_h/d_w = d_s/2$.

Theorem (Steinhurst-Teplyaev '10)

$s \mapsto \zeta_\Delta(s,\gamma)$ admits a meromorphic extension to $\mathbb{C}$.

(Poles of $\zeta_\Delta(s,0)$, for GSC $\subset \mathbb{R}^2$)



$$d_{k,p} := 2\left(\frac{d_k}{d_w} + \frac{2p\pi i}{\log(m_F \rho_F)}\right)$$

The $d_{k,p}$ correspond to the complex dims of the fractal, à la M. Lapidus.

$$\mathrm{Res}\left(\zeta_\Delta(\cdot,0), \frac{d_{k,p}}{2}\right) = \frac{\hat{G}_{k,p}}{\Gamma\left(d_{k,p}/2\right)}.$$

Moreover, $\zeta_\Delta(s,0) = 0 \quad \forall s \in -\mathbb{N}$.

Atomic gas in Sierpinski carpets $(H = -\Delta, \mu \leq \inf \mathrm{Spec}(H))$

$$\log \Xi_{\beta,\mu} = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \beta^{-t} \Gamma(t) \zeta(t+1) \zeta_\Delta(t, -\mu) dt, \quad \sigma > d_s/2.$$

Particle density should be the number of bosons per unit spectral volume ρ.
To take the infinite volume limit, we exhaust the unbounded carpet $F_\infty = \bigcup_{n=0}^\infty I_F^n F$ by an increasing family of carpets $\{\Lambda_n\}_n = \{I_F^n F\}_n$.

$$\rho_{\Lambda_n}(\beta, \mu) := \omega_{\Lambda_n, \beta, \mu} \left(\frac{\mathbf{N}}{V_s(\Lambda_n)} \right) = \frac{1}{V_s(\Lambda_n)} \mathrm{Tr} \frac{1}{e^{\beta(-\Delta_{\Lambda_n} - \mu)} - 1} = \frac{1}{V_s(\Lambda_n)} \sum_{j=0}^\infty \frac{1}{e^{\beta(E_j(\Lambda_n) - \mu)} - 1}.$$

$E_j(\Lambda_n)$ is the $(j+1)$th eigenvalue of the Laplacian $-\Delta_{\Lambda_n}$.

Lemma

As $n \to \infty$, the density of Bose gas in Λ_n at (β, μ) is

$$\rho_{\Lambda_n}(\beta, \mu) = \frac{1}{(4\pi\beta)^{d_s/2} \hat{G}_{0,0}} \sum_{m=1}^\infty e^{m\beta\mu} G_0 \left(-\log \left(\frac{m\beta}{(I_F)^{2n}} \right) \right) m^{-d_s/2} + o(1).$$

In particular, the upper critical density $\overline{\rho}_c(\beta) := \limsup_{n\to\infty} \rho_{\Lambda_n}(\beta, 0) < \infty$ iff $d_s > 2$, in which case

$$\overline{\rho}_c(\beta) = \frac{\max(G_0)}{(4\pi\beta)^{d_s/2} \hat{G}_{0,0}} \zeta\left(\frac{d_s}{2} \right).$$

Bose-Einstein condensation in Sierpinski carpets

Theorem (C.)

Assume $d_s > 2$. For each $\rho_{\rm tot} > 0$, let μ_n be the unique root of $\rho_{\Lambda_n}(\beta, \mu_n) = \rho_{\rm tot}$.

1. If $\rho_{\rm tot} \le \overline{\rho}_c(\beta)$, and $\overline{\mu}$ is the root of $\limsup_{n\to\infty} \rho_{\Lambda_n}(\beta, \overline{\mu}) = \rho_{\rm tot}$, then $\liminf_{n\to\infty} \mu_n = \overline{\mu}$.
2. If $\rho_{\rm tot} > \overline{\rho}_c(\beta)$, then $\lim_{n\to\infty} \mu_n = 0$. Moreover if we denote the boson occupation density in the ground state, a.k.a. the condensate density, by

$$\rho^0_{\Lambda_n}(\beta, \mu) := \frac{1}{V_s(\Lambda_n)} \frac{1}{e^{\beta(E_0(\Lambda_n)-\mu)} - 1},$$

then $\lim_{n\to\infty} \left[\rho^0_{\Lambda_n}(\beta, \mu_n) + \rho_{\Lambda_n}(\beta, 0)\right] = \rho_{\rm tot}$.

Remark. A similar analysis shows that the liminf free energy density, regarded as a function of particle density, is non-analytic at $\overline{\rho}_c(\beta)$.

Interpretation

If the Bose gas density exceeds $\overline{\rho}_c(\beta)$, any excess density must condense in the lowest eigenfunction of the Laplacian. This is the fractal version of Bose-Einstein condensation (BEC), a *quantum phase transition*.
(unique KMS state in regime (1), non-unique KMS states in regime (2)).

Criterion for BEC in Sierpinski carpets

Theorem (C.)

For an unbounded GSC, the following are equivalent:

1. Spectral dimension $d_s > 2$.
2. (The Brownian motion whose generator is) the Laplacian is transient.
3. BEC exists for a low-temperature, high-density ideal Bose gas.

Transience of BM $\Leftrightarrow$ BEC of ideal Bose gas in unbounded spaces appears to be a very general principle. It has been established on:

- Euclidean spaces: $\mathbb{R}^d$, $\mathbb{Z}^d$.
- Inhomogeneous graphs: Comb graphs, density-0 perturbations of Cayley trees. [Fidaleo-Guido-Isola, Matsui]
- Fractals: GSCs.

Looking ahead

- **Existence of BEC for interacting Bose gas on fractals?**
  More conveniently studied on GSC graphs.
  Special case: Hardcore Bose gas, where interactions take place on single site and no more than 1 particle occupies each site. Then the Hamiltonian can be mapped 1:1 to that of the (spin-$\frac{1}{2}$) XY-model. Existence of BEC $\Leftrightarrow$ Existence of phase transition in quantum XY. Also useful to consider classical XY.
  - **Thm**. Recurrence of SRW $\Rightarrow$ No p.t. in XY. [Cassi '92]
  - Generally, transience of SRW is necessary, but not sufficient, for p.t. in XY. [Pemantle-Steif '99, Häggström '00].
  - **Problem:** Show on GSC graphs whether transience of SRW is sufficient for p.t. in XY. [On $\mathbb{Z}^d$ this was answered by Fröhlich-Simon-Spencer '76 (classical XY) and Dyson-Lieb-Simon '78 (quantum XY).]
- **Dynamics of dilute Bose gas on fractals? [superfluidity, solitons]**
  In $\mathbb{R}^d$ it has been proved that the mean-field Gross-Pitaevskii eqn, a cubic nonlinear Schrödinger eqn, is the correct governing PDE in the scaling limit where the particle number $\to \infty$. [Erdös-Schlein-Yau '06] Is this the case on Sierpinski carpets?
- **Developing some notion of field theory on fractals may be fruitful.**
  Study Gaussian free fields (GFFs): $\mathbb{E}\phi_x = 0$, $\mathbb{E}[\phi_x \phi_y] = G(x,y)$, $\forall x, y \in F$.
  In joint work with Baris Ugurcan (to appear later in '12), we have:
  - Given explicit Lévy-type construction of GFFs on post-critically finite (pcf) fractals (e.g. Sierpinski gaskets). and described their regularity properties.
  - Similarly characterized GFFs on GSCs: GFFs diverge at every point a.s. if $d_s \geq 2$.
  - Computed the expected maxima of GFFs on both pcf fractals and GSCs. Two consequences of this result: 1) Obtained sharp asymptotics of the cover times on fractal graphs; 2) Established the chaos (or superconcentration, à la Sourav Chatterjee) of GFFs on fractals with $d_s \geq 2$.