Schrödinger operators with decaying oscillatory potentials

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Schrödinger operators on a half-line

- We investigate half-line Schrödinger operators $H = -\Delta + V$
- 0 will be a regular endpoint (and most of our statements will be independent of boundary condition)
- V should obey an oscillation condition and decay at $+\infty$
- What can we say about the spectrum of H?
- (Corollary of Weyl's theorem for relatively compact perturbations) If V decays at infinity, in the sense that

$$\lim_{n\to\infty} \int_n^{n+1} |V(x)| dx = 0$$

  then $\sigma_{\mathrm{ess}}(H) = [0, +\infty)$
- What about the decomposition into absolutely continuous, singular continuous, pure point spectrum? How stable is a.c. spectrum?

Decaying potentials with harmonic oscillations

- (the Wigner–von Neumann potential)
  Explicit potential on $(0, +\infty)$ with asymptotic behavior
  $$V(x) = -8\frac{\sin 2x}{x} + O(x^{-2}), \quad x \to \infty$$
  such that $-\Delta + V$ has eigenvalue $+1$ embedded in the a.c. spectrum $(0, +\infty)$
- (Atkinson, Harris–Lutz, Ben Artzi–Devinatz)
  If
  $$V(x) = \sum_{k=1}^{K} \lambda_k \frac{\sin(\alpha_k x)}{x^{\gamma_k}} + W(x)$$
  with $\gamma_k > \frac{1}{2}$ and $W \in L^1$, then H has purely a.c. spectrum on $(0, +\infty) \setminus \left\{ \frac{\alpha_k^2}{4} \mid 1 \le k \le K \right\}$
- (Weidmann)
  If V has bounded variation and $V(x) \to 0$ as $x \to +\infty$, then $-\Delta + V$ has purely a.c. spectrum on $(0, +\infty)$

Generalized bounded variation: definition

- A function $\beta(x)$ has rotated bounded variation with phase ϕ if $e^{i\phi x}\beta(x)$ has bounded variation
- A function $V(x)$ has generalized bounded variation with phases $\phi_1, \ldots, \phi_L$ ($L < \infty$) if

$$V(x) = \sum_{l=1}^{L} \beta_l(x) + W(x)$$

where β_l has rotated bounded variation with phase ϕ_l and $W \in L^1$

- Example of rotated bounded variation:

$$\frac{e^{-i(\phi x+\alpha)}}{(1+x)^\gamma}, \quad \text{with } \gamma > 0$$

- Example of generalized bounded variation:

$$\frac{\cos(\phi x+\alpha)}{(1+x)^\gamma}, \quad \text{with } \gamma > 0$$

or a linear combination of such terms

Potentials of generalized bounded variation

Theorem 1. *Let* $V : (0, \infty) \to \mathbb{R}$ *be such that*

- V *has generalized bounded variation with set of phases* $A = \{\phi_1, \dots, \phi_L\}$, *i.e.*

$$V(x) = \sum_{l=1}^{L} \beta_l(x) + W(x)$$

where $e^{i\phi_l x}\beta_l(x)$ *has bounded variation and* $W \in L^1$

- $V \in L^1 + L^p$ *for some* $p < \infty$

Then the operator $H = -\Delta + V$ *on* $L^2(0, +\infty)$ *satisfies*

- $\sigma_{\mathrm{ac}}(H) = [0, +\infty)$
- $\sigma_{\mathrm{sc}}(H) = \emptyset$
- $\sigma_{\mathrm{pp}}(H) \cap (0, \infty) \subset \left\{ \frac{\eta^2}{4} \;\middle|\; \eta \in \bigcup_{k=1}^{p-1} (\underbrace{A + \cdots + A}_{k \text{ times}}) \right\}$ *is a finite set*

Application: slowly decaying Wigner–von Neumann type potentials

$$V(x) = \sum_{k=1}^{K} \lambda_k \frac{\cos(\alpha_k x + \xi_k)}{x^{\gamma_k}} + W(x), \qquad \gamma_k > 0, \quad W \in L^1$$

Existence of embedded eigenvalues

Theorem 2. *For a generic choice of* $\alpha_1, \ldots, \alpha_K$ *and*

$$E = \frac{1}{4}(\pm\alpha_{j_1} \pm \cdots \pm \alpha_{j_{p-1}})^2,$$

there exists an L^p *potential of generalized bounded variation*

$$V(x) = \sum_{k=1}^{K} \lambda_k \frac{1}{x^\gamma} \cos(\alpha_k x + \xi_k(x)) + \beta_0(x), \qquad x \geq x_0$$

such that $-\Delta + V$ *has a real-valued eigenfunction* $u(x)$ *at energy* E *with asymptotics*

$$\frac{1}{\sqrt{E}} u'(x) + iu(x) = Af(x)e^{i[\sqrt{E}x + \theta_\infty]}(1 + o(1)), \quad x \to \infty$$

with

$$f(x) = \begin{cases} x^{-C\lambda_{j_1} \ldots \lambda_{j_{p-1}}} & \gamma = \frac{1}{p-1} \\ \exp\left(-\frac{C}{1-(p-1)\gamma} \lambda_{j_1} \ldots \lambda_{j_{p-1}} x^{1-(p-1)\gamma}\right) & \gamma \in (\frac{1}{p}, \frac{1}{p-1}) \end{cases}$$

and $A, C > 0$

Extending to infinitely many frequencies

- Our Theorem 1 applies, in particular, to potentials of the form

$$V(x) = \tau(x) \sum_{k=1}^{K} c_k e^{i\phi_k x}$$

where $K < \infty$ and $\tau(x) \in L^p$ has bounded variation

- Can we generalize to potentials of the form

$$V(x) = \tau(x) W(x)$$

where $W(x)$ is a more general oscillatory function?

Periodic potential $\times$ decay

Theorem 3. *Let $V(x) = \tau(x)W(x)$, where*

- $\tau(x) \in L^p$ *has bounded variation*
- $W(x)$ *is periodic of period T, such that*
    - *the Fourier series of W converges in $L^1(0, T)$ to W*
    - *the Fourier coefficients of W obey $\sum_{n \in \mathbb{Z} \setminus \{0\}} \left| \frac{\hat{W}_n}{n} \right| < \infty$*

  (for instance, let $W \in L^2(0, T)$)

Then

- $\sigma_{\mathrm{ac}}(H) = [0, +\infty)$
- $\sigma_{\mathrm{sc}}(H) = \emptyset$
- $\sigma_{\mathrm{pp}}(H) \cap (0, \infty) \subset \left\{ \frac{k^2 \pi^2}{T^2} \;\middle|\; k \in \mathbb{Z} \right\}$ *is at most countable*

Almost periodic potential $\times$ decay

Theorem 4. *Let* $V(x) = \tau(x)W(x)$*, where*

- $\tau(x) \in L^p$ *has bounded variation*
- $W(x) = \sum_{k=1}^{\infty} c_k e^{i\phi_k x}$*, with* $\sum |c_k|^\alpha < \infty$ *for some* $\alpha \in [0, 1)$

Then there is a set S*, independent of boundary condition at* 0*, which supports the singular part of the spectral measure, such that*

$$\dim_H S \leq (p-1)\alpha$$

($\dim_H$ *stands for Hausdorff dimension) and* $\sigma_{\mathrm{ac}}(H) = [0, +\infty)$*.*

All the theorems stated have analogs for orthogonal polynomials on the real line and unit circle

Thank you for your attention!