Absolutely continuous spectrum for random Schrödinger operators on tree-strips of finite cone type.

Christian Sadel

University of California, Irvine

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- A tree-strip of finite cone type of width m is the product graph of a finite graph $\mathbb{G}$ with m vertices and a rooted tree of finite cone type.
- The product graph of two graphs $\mathbb{G}_1$ and $\mathbb{G}_2$ is defined as follows: (x_1, y_1) and (x_2, y_2) are connected by an edge, iff either $x_1 = x_2$ and y_1 and y_2 are connected by an edge, or $y_1 = y_2$ and x_1 and x_2 are connected by an edge.

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- A tree of finite cone type can be constructed from an $s \times s$ substitution matrix S with non-negative integer entries, as follows: Each vertex has a label $p \in \{1, \ldots, s\}$. A vertex of label p has $S_{p,q}$ children of label q. Except for the root, a vertex of label p has $\sum_q S_{p,q} + 1$ neighbors. The tree is determined by the label of the root, if the label is p we call the tree $\mathbb{T}^{(p)}$.
- The cone of descendents of a vertex with label p is equal to $\mathbb{T}^{(p)}$. Hence, there are only finitely many different types of cones of descendants and therefore the tree is called of 'finite cone type'.

- We consider the Hilbert space $\ell^2(\mathbb{T}^{(p)} \times \mathbb{G}) = \ell^2(\mathbb{T}^{(p)}) \otimes \mathbb{C}^m = \ell^2(\mathbb{T}^{(p)}, \mathbb{C}^m)$. An element $u \in \ell^2(\mathbb{T}^{(p)}, \mathbb{C}^m)$ is considered as a function $u : \mathbb{T}^{(p)} \mapsto \mathbb{C}^m$ with $\sum_{x \in \mathbb{T}^{(p)}} \|u(x)\|^2 < \infty$.
- On $\ell^2(\mathbb{T}^{(p)}, \mathbb{C}^m)$ we consider the random Schrödinger operators

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- $V(x) \in \mathrm{Sym}(m)$ for $x \in \mathbb{B}$ are i.i.d. random, real symmetric matrices with common distribution μ. We assume that all mixed moments of μ exist. We may think of $V(x)$ as random matrix-valued potential.
- $\lambda \in \mathbb{R}$ is the disorder and supposed to be small.
- One may write $H_\lambda = \Delta \otimes 1 \,+\, 1 \otimes A \,+\, \lambda \bigoplus_{x \in \mathbb{T}^{(p)}} V(x)$.

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- With some weaker assumptions, Keller, Lenz and Warzel showed that the spectrum of the adjacency operator on $\mathbb{T}^{(p)}$ is purely a.c., independant on p and consists of finitely many intervals. Let us call this set Σ.

- We denote the eigenvalues of the free vertical operator A by $a_1 \leq a_2 \leq \ldots \leq a_m$.
- There is an orthogonal matrix $O \in O(m)$ such that $O^t A O = \mathrm{diag}(a_1, \ldots, a_m) = A_d$.
- Let $\mathcal{U} u(x) = O u(x)$ then $\mathcal{U}$ is unitary and

$$\mathcal{U}^* H_\lambda \mathcal{U} \;=\; \Delta \otimes 1 + 1 \otimes A_d + \lambda \bigoplus_{x \in \mathbb{B}} O^t V(x) O\,.$$

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- Hence by changing the distribution μ to $O^t \mu O$ we may assume w.l.o.g. that $A = A_d$ is diagonal.
- The free operator can be written as direct sum

$$H_0 \;=\; \Delta \otimes 1 + 1 \otimes A_d \;=\; \bigoplus_{k=1}^{m} \Delta \;+\; a_k$$

and has a.c. band spectrum $\bigcup_k (\Sigma + a_k)$.

Main Theorems

- Assume S and A are such that $I_{A,S} = \bigcap_k (\Sigma + a_k)$ has non-empty interior.
  (For fixed A and S there exists a b_0 such that for $b > b_0$ the set $I_{A,bS}$ has non-empty interior).

Theorem (S. 2012)

Under the assumptins above, there exists a dense open subset $I \subset I_{A,S}$ and an open neighborhood U of $\{0\} \times \mathbb{R}$ in $\mathbb{R}^2$ such that one obtains the following:
The spectrum of H_λ in $U_\lambda = \{E : (\lambda, E) \in U\}$ is purely absolutely continuous with probability one.

Some previous results:

- Klein (1994, 1996) a.c. spectrum and ballistic wave spreading for Anderson model on Bethe lattice (regular tree)
- Aizenman, Sims, Warzel (2006), different proof for ac spectrum
- Froese, Hasler, Spitzer (2007), different proof for ac spectrum
- Keller, Lenz, Warzel (2010), ac spectrum for Anderson model on trees of finite cone type
- Froese, Halasan, Hasler (2010), ac spectrum for Bethe strip of width 2
- Klein, S. (2011) a.c. spectrum and ballistic behavior for Bethe strip of arbitrary finite width
- Aizenman, Warzel (2011), a.c. spectrum for Anderson model on Bethe lattice for small disorder in bigger region; absence of mobility edge for small disorder; ballistic behavior in this regime

- The $m \times m$ Green's matrix function of H_λ is given by

$$[G_\lambda(x, y; z)]_{k,l} = \langle x, k | (H_\lambda - z)^{-1} | y, l \rangle$$

for $x, y \in \mathbb{T}^{(p)}$, $k, l \in \mathbb{G}$ and $z = E + i\eta$ with $E \in \mathbb{R}$, $\eta > 0$.

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- We define the integrated, averaged density of states by

$$N_\lambda(E) = \tfrac{1}{m}\,\mathbb{E}\left(\sum_{k=1}^{m} \langle 0,k|\chi_{(-\infty,E]}(H_\lambda)|0,k\rangle\right) \quad \text{for the root } 0 \in \mathbb{T}^{(p)}$$

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Theorem

There is an open neihborhood U of $\{0\} \times I$ as above, such that the following hold: The integrated, averaged density of states is absolutely continuous in U_λ, its density is strictly positive in U_λ and it depends continuously on $(\lambda, E) \in U$.

The Theorems above follow from the following:

Theorem

There is a dense open subset $I \subset I_{A,S}$ *and an open neighborhood* U *of* $\{0\} \times I$ *in* $\mathbb{R}^2$ *, such that for all* $x \in \mathbb{T}^{(p)}$ *the continuous functions*

$$(\lambda, E', \eta) \in U \times (0, \infty) \longrightarrow \mathbb{E}(|G_\lambda(x, x; E' + i\eta)|^2) \tag{1}$$

and

$$(\lambda, E', \eta) \in U \times (0, \infty) \longrightarrow \mathbb{E}(G_\lambda(x, x; E' + i\eta)) \tag{2}$$

have continuous extensions to $U \times [0, \infty)$.

Idea of Proof

- Let G be a symmetric $m \times m$ matrix with positive imaginary part.
- On the set of pairs of positive semi-definite matrices M we define the exponential decaying functions

$$f_G(M_+, M_-) = \exp(i\mathrm{Tr}(GM_+ - G^* M_-)) .$$

- Let $G_\lambda^{(p)}(z)$ denote the matrix Green's function at the root of $\mathbb{T}^{(p)}$ for the random operator H_λ and define

$$\xi^{(p)}_{\lambda,E,\eta} = \mathbb{E}\left(f_{G_\lambda^{(p)}(E+i\eta)} \right)$$

- There is a certain Banach space $\mathcal{K}$, such that $\left[\xi^{(p)}_{\lambda,E,\eta}\right]_p$ is in this Banach space for $\mathrm{Im}\,(z) > 0$ and it converges in $\mathcal{K}$ for $\lambda = 0$, $E \in \mathring{I}_{A,S}$ and $\eta \downarrow 0$.

- Moreover, there is a continuous function $F : \mathbb{R}^2 \times (0, \infty) \times \mathcal{K} \to \mathcal{K}$, Frechet differentiable with respect to the element in $\mathcal{K}$, such that

$$F\left(\lambda, z, [\xi^{(p)}_{\lambda,z}]_p\right) = 0 \; .$$

- Use the implicit function Theorem at $\lambda = 0$, $E \in I \subset I_{A,S}$ to show that $[\xi^{(p)}_{\lambda,E+i\eta}]_p$ extends continuously to $(\lambda, E, \eta) \in U \times [0, \infty)$ where U is an open neighborhood of I which in turn is a dense open subset of the interior of $I_{A,S}$.
- Everything follows from these continuous extensions.