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Derivation of effective evolution equations from quantum dynamics

References:

- Proceedings of conference "Days in PDEs", 2011, arxiv: 1111.6995
- Lecture Notes for summer school of Clay Mathematical Institute, 2008: arxiv: 0807.4307
- For central limit theorem, look at recent paper: arxiv: 1111.6999

Following pages: The Gross-Pitaevskii regime.

Scattering length defined through

$$\left(-\Delta + \tfrac{1}{2} V\right) f = 0, \quad f \xrightarrow[|x| \to \infty]{} 1$$

$\Rightarrow$ $f(x) \simeq 1 - \dfrac{a_0}{|x|}$ for $|x|$ large

Equivalently: $\quad 8\pi a_0 = \int dx\, V(x) f(x)$

Then:

$$\left(-\Delta + \tfrac{1}{2} N^2 V(Nx)\right) f(Nx) = 0$$

$$f(Nx) \simeq 1 - \frac{a_0}{N|x|} = 1 - \frac{(a_0/N)}{|x|}$$

$\Rightarrow$ scattering length of $N^2 V(Nx)$ is $\dfrac{a_0}{N}$.

The Gross-Pitaevskii regime

Study dynamics generated by

$$H_N = \sum_{j=1}^{N} -\Delta_{x_j} + \sum_{i<j}^{N} N^2 V(N(x_i - x_j))$$

where $V \geq 0$, short range, with scattering length a_0.

Theorem: let $\Psi_N \in L^2(\mathbb{R}^{3N})$ be s.t.

- $\langle \Psi_N, H_N \Psi_N \rangle \leq C \cdot N$
- $\gamma_N^{(1)} \longrightarrow |\varphi\rangle\langle\varphi|$ for some $\varphi \in L^2(\mathbb{R}^3)$

Then $\forall$ fixed $k \in \mathbb{N}$, $t \in \mathbb{R}$

$$\gamma_{N,t}^{(k)} \xrightarrow[N\to\infty]{} |\varphi_t\rangle\langle\varphi_t|^{\otimes k}$$

where

$$i\partial_t \varphi_t = -\Delta\varphi_t + 8\pi a_0 |\varphi_t|^2 \varphi_t$$

with $\varphi_{t=0} = \varphi$.

Connection with mean-field regime:

We can write

$$H_N = \sum_{j=1}^{N} -\Delta_{x_j} + \frac{1}{N} \cdot \sum_{i<j}^{N} N^3 V(N(x_i - x_j))$$

where

$$N^3 V(N(x_i - x_j)) \longrightarrow b_0 \cdot \delta(x_i - x_j)$$

with $\quad b_0 = \int dx\, V(x)$.

Analogy explains local nonlinearity in GP equation but it does not explain emergence of scattering length!

Reason: physics is very different, and correlations among particles are here crucial.

Consequence: general strategy of proof is still valid, but several modifications are required.

Proof of convergence: consider evolution of 1-particle density:

$$i\partial_t \gamma_{N,t}^{(1)} = [-\Delta, \gamma_{N,t}^{(1)}] + (N-1)\,\mathrm{tr}_2 [N^2 V(N(x_1-x_2)), \gamma_{N,t}^{(2)}]$$

We have

$$(N-1)\,\mathrm{tr}_2\, N^2 V(N(x_1-x_2))\cdot\gamma_{N,t}^{(2)} \cong$$

$$\cong \int dx_2\, N^3 V(N(x_1-x_2))\cdot \gamma_{N,t}^{(2)}(x_1,x_2;x_1',x_2)$$

Suppose $\gamma_{N,t}^{(2)} \longrightarrow \gamma_{\infty,t}^{(2)}$ as $N\to\infty$. Then we may expect

$$\gamma_{N,t}^{(2)}(x_1,x_2;x_1',x_2) \simeq$$

$$\simeq f(N(x_1-x_2))\, f(N(x_1'-x_2))\cdot \gamma_{N,t}^{(2)}(x_1,x_2;x_1',x_2)$$

Hence

$$(N-1)\,\mathrm{tr}_2\, N^2 V(N(x_1-x_2))\cdot\gamma_{N,t}^{(2)} \cong$$

$$\cong \int dx_2\, N^3 V(N(x_1-x_2)) f(N(x_1-x_2))\cdot \gamma_{\infty,t}^{(2)}(x_1,x_2;x_1',x_2)$$

$$\cong 8\pi a_0\, \gamma_{\infty,t}^{(2)}(x_1,x_1;x_1',x_1) = 8\pi a_0\, \mathrm{tr}_2\, \delta(x_1-x_2)\gamma_{\infty,t}^{(2)}$$

Extending to all $K \geq 1$, we find that every limit point $\{\gamma_{\infty,t}^{(k)}\}_{k\geq 1}$ solves the ∞-hierarchy:

$$i\partial_t \gamma_{\infty,t}^{(k)} = \sum_{j=1}^{k} \left[-\Delta_{x_j}, \gamma_{\infty,t}^{(k)}\right]$$
$$+ 8\pi a_0 \sum_{j=1}^{k} \mathrm{tr}_{k+1} \left[\delta(x_j - x_{k+1}), \gamma_{\infty,t}^{(k+1)}\right]$$

<u>Main challenge</u>: show that short scale correlation structure can be described by $f(N(x_i - x_j))$.

To this end, we use a-priori bounds

$$\int d\vec{x} \left| \nabla_1 \nabla_2 \frac{\Psi_{N,t}}{f(N(x_1-x_2))} \right|^2 \leq C$$

uniformly in N.

Note: $\left.\begin{array}{l} \int d\vec{x} \, |\nabla_1 \nabla_2 \Psi_{N,t}|^2 \simeq N \\ \int dx \, |\nabla^2 f(Nx)|^2 \simeq N \end{array}\right\}$ big cancellations!

Proof of uniqueness: two new obstacles compared with Coulomb case.

- Due to correlations, the bound

$$\langle \psi_{N,t}, (1-\Delta_1)\cdots(1-\Delta_k)\psi_{N,t}\rangle$$
$$= \text{tr } (1-\Delta_1)\cdots(1-\Delta_k)\,\gamma_{N,t}^{(k)} \leq C^k$$

cannot hold uniformly in N.

Solution: prove

$$\langle \psi_{N,t}, \nabla_1^* \cdots \nabla_k^* \, \chi_1(\vec{x})\cdots\chi_k(\vec{x})\, \nabla_k \cdots \nabla_1 \psi_{N,t}\rangle \leq C^k$$

with
$$\chi_j(\vec{x}) \simeq \begin{cases} 1 & \text{if } |x_j - x_i| \geq \ell \ \forall i \\ 0 & \text{otherwise} \end{cases}$$

Choosing $\frac{1}{N} \ll \ell \ll 1$ appropriately, we conclude that

$$\text{tr } (1-\Delta_1)\cdots(1-\Delta_k)\,\gamma_{N,t}^{(k)} \leq C^k$$

• In 3-dim

$$\delta(x) \not\leq C\,(1-\Delta) \qquad \left(\begin{array}{c}\text{because} \\ \|\varphi\|_\infty \not\leq C \cdot \|\varphi\|_{H^1}\end{array}\right)$$

<u>Consequence</u>: a-priori estimates are not enough to control the singularity of the interaction.

<u>Solution</u>: to bound error term

$$\int_0^t ds_1 \cdots \int_0^{s_{n-1}} ds_n \; \mathcal{U}^{(k)}(t-s_1)\, B^{(k)} \cdots B^{(k+n-1)} \gamma^{(k+n)}_{\infty, s_n}$$

need to use smoothing effect of free evolution to show

$$\|\text{error}\|_{H^1_k} \leq \left(C\, t^{1/2}\right)^n$$

and conclude uniqueness.

<u>Method</u>: reorganize factors using diagrammatic expansion in terms of Feynman graphs.