A Gordon type theorem for measure perturbed Schrödinger operators

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Motivation

Three principles in modelling quasicrystals:

- locally regular/ordered
- globally aperiodic
- randomization

1976 Gordon: $H = -\frac{d^2}{dx^2} + V$ does not have eigenvalues for "nice" $V \in L_\infty(\mathbb{R})$
2000 Damanik/Stolz: same result for $V \in L_{1,\mathrm{loc}}(\mathbb{R})$
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Gordon measures

Definition

Let $\mu = \mu_+ - \mu_-$ be a signed Borel measure.
μ *uniformly locally bounded* ($\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$)

$$:\Longleftrightarrow \|\mu\|_{\mathrm{loc}} := \sup_{x\in\mathbb{R}} |\mu|\left([x, x+1]\right) < \infty.$$

μ *Gordon measure* $:\Longleftrightarrow \mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ and there exists (μ^m) in $\mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R})$ of periodic measures with period sequence (p_m): $p_m \to \infty$ and

$$\forall\, C \geq 0 : \lim_{m\to\infty} e^{Cp_m} |\mu - \mu^m|\left([-p_m, 2p_m]\right) = 0.$$

Definition of H_μ: form methods

$\mu \in \mathcal{M}_{\mathrm{loc,unif}}(\mathbb{R}) \implies \mu$ infinitesimally form small with respect to classical Dirichlet form

$$D(\tau_0) := W_2^1(\mathbb{R}), \quad \tau_0(u,v) := \int u'\overline{v'},$$

i.e.

$$\forall \gamma \in (0,1)\, \exists C_\gamma \geq 0 : \int |u|^2 \, d\mu \leq \gamma \tau_0(u,u) + C_\gamma \|u\|_{L_2(\mathbb{R})}^2 \quad (u \in D(\tau_0)).$$

Hence:

$$D(\tau_\mu) := W_2^1(\mathbb{R}), \quad \tau_\mu(u,v) := \int u'\overline{v'} + \int u\overline{v}\, d\mu$$

densely defined, symmetric, semibounded from below and closed. Let $H_\mu \sim \tau_\mu$, i.e.

$$(H_\mu u \,|\, v) = \tau_\mu(u,v) \quad (u \in D(H_\mu), v \in D(\tau_\mu)).$$

Gordons Theorem

Theorem (S 2011)

Let μ be a Gordon measure. Then H_μ does not have any eigenvalues.

Proof

Fix normalized initial condition at 0. Let u be the solution of $H_\mu u = Eu$, u_m the solution of $H_{\mu^m} u_m = Eu_m$ ($m \in \mathbb{N}$). By Gronwall inequality:

$$\left\| \begin{pmatrix} u(x) \\ u'(x+) \end{pmatrix} - \begin{pmatrix} u_m(x) \\ u_m'(x+) \end{pmatrix} \right\| \leq Ce^{C|x|} \left|\mu - \mu^m\right|([0,x])$$
$$\leq \frac{1}{4} \quad (x \in [-p_m, 2p_m],\ m \text{ large}).$$

For solutions v to periodic measures with period p:

$$\max\left\{ \left\| \begin{pmatrix} v(-p) \\ v'(-p+) \end{pmatrix} \right\|, \left\| \begin{pmatrix} v(p) \\ v'(p+) \end{pmatrix} \right\|, \left\| \begin{pmatrix} v(2p) \\ v'(2p+) \end{pmatrix} \right\| \right\} \geq \frac{1}{2}.$$

Hence,

$$\limsup_{|x|\to\infty} \left(|u(x)|^2 + \left|u'(x+)\right|^2 \right) \geq \left(\frac{1}{4}\right)^2 > 0.$$

Therefore, $u \notin D(H_\mu)$.

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