Bounds on fluctuations for Mallows random permutations: Arizona School of Analysis and Mathematical Physics

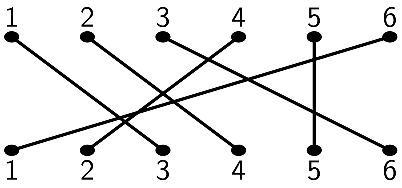
Shannon Starr

University of Rochester

March 14, 2012

- Thermodynamic Limit for the Mallows Model on S_n. *J. Math. Phys.* 2009
- The length of the longest increasing subsequence of a random Mallows permutation. *J. Theoret. Probab.* 2011 (to appear) joint work with **Carl Mueller**, UR
- and joint work with **Meg Walters**, UR, in preparation.

What is a Mallows random permutation?



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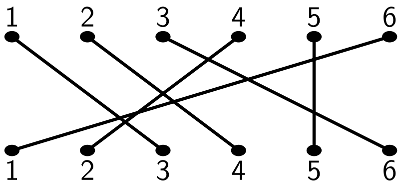

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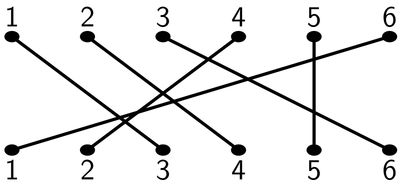

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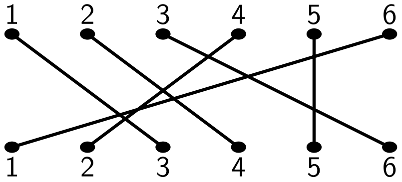

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- Can define a classical Hamiltonian on S_n:

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$$A(\beta) = \int_0^1 \ln\left(\frac{1-e^{-\beta x}}{\beta x}\right) dx\,, \qquad B(\beta) = \sqrt{\frac{e^{\beta}-1}{\beta}}.$$

A weak limit law

Example:

$\pi_1 = 3$
$\pi_2 = 4$
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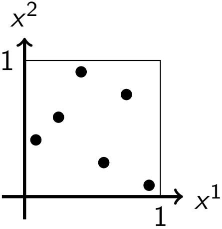

Empirical measure on $[0,1]^2$

$$\hat{\rho}_{n,\pi} = \frac{1}{n}\sum_{i=1}^{n}\delta_{(i/n,\,\pi_i/n)}$$

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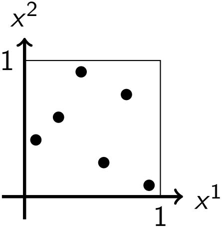

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Theorem. For $\beta \in \mathbb{R}$ fixed, take $q_n(\beta) = \exp(-\beta/n)$.
There exists a density ρ_β on $[0,1]^2$ such that,
for any continuous function φ on $[0,1]^2$,

$$\mu_{n,q_n(\beta)} \left\{ \pi \in S_n \, : \, \left| \int \varphi \, d\hat{\rho}_{n,\pi} - \int \varphi \, d\rho_\beta \right| > \epsilon \right\} \to 0 \quad \text{as } n \to \infty,$$

for each fixed $\epsilon > 0$.

Denote: $\mathbf{x} = (x^1, x^2) \in [0,1]^2$.
Boltzmann-Gibbs measure on $([0,1]^2)^n$:

$$d\mu_{n,\beta}(\mathbf{x}_1,\dots,\mathbf{x}_n) \,=\, \frac{e^{-\beta H_n(\mathbf{x}_1,\dots,\mathbf{x}_n)}}{Z_n(\beta)}\, d\mathbf{x}_1 \cdots d\mathbf{x}_n\,,$$

$$H_n(\mathbf{x}_1,\dots,\mathbf{x}_n) \,=\, \frac{1}{n} \sum_{1\leq i<j\leq n} h(\mathbf{x}_i,\mathbf{x}_j)\,,$$

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Then ρ_β is the unique measure on $[0,1]^2$ satisfying

$$\frac{d\rho_\beta(\mathbf{x})}{d\mathbf{x}} \,=\, \frac{1}{\mathcal{Z}(\beta)} \exp\left(-\beta \int_{[0,1]^2} h(\mathbf{x},\mathbf{x}')\, d\rho_\beta(\mathbf{x}')\right)$$

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$$= \frac{(\beta/2)\sinh(\beta/2)}{\left(e^{\beta/4}\cosh(\frac{\beta}{2}[x-y]) - e^{-\beta/4}\cosh(\frac{\beta}{2}[x+y-1])\right)^2}\,.$$

6/18. Length of the Longest Increasing Subsequence

For $\pi \in S_n$,

$$L_n(\pi) = \max\{k \leq n : \exists i_1 < \cdots < i_k \text{ s.t. } \pi_{i_1} < \cdots < \pi_{i_k}\}.$$

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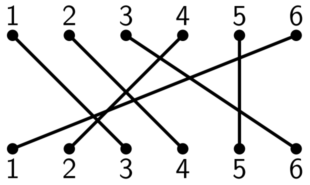

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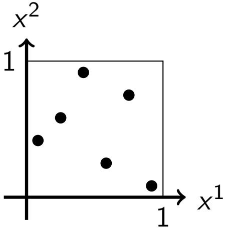

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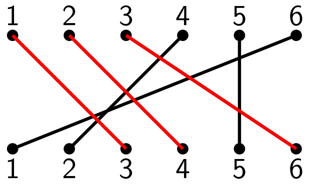

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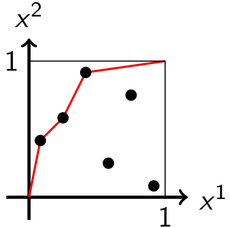

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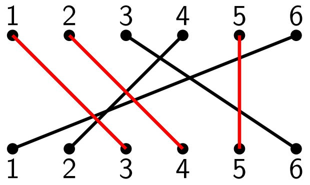

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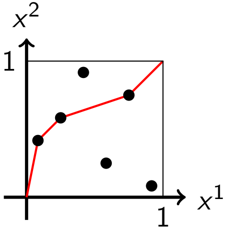

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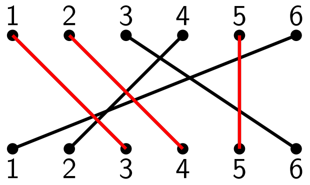

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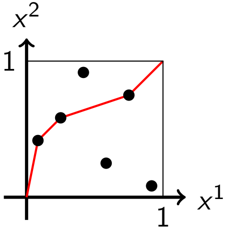

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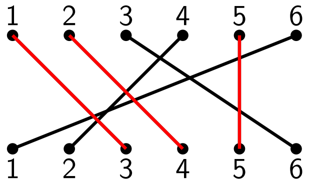

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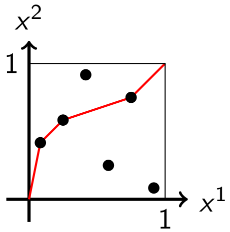

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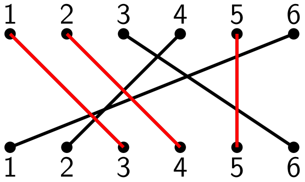

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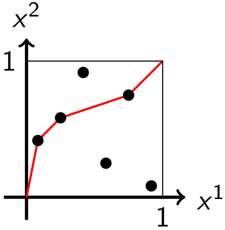

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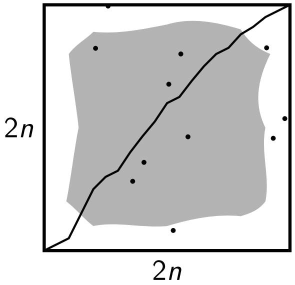
For the uniform measure μ_n on S_n ($\beta = 0$),

$$\lim_{n\to\infty} \mu_n \left\{\pi \,:\, |n^{-1/2} L_n(\pi) - 2| > \epsilon\right\} \;=\; 0\,,$$

for all $\epsilon > 0$.

 Hammersley's proof: $n^{-1/2}\mathbb{E}L_n$ converges



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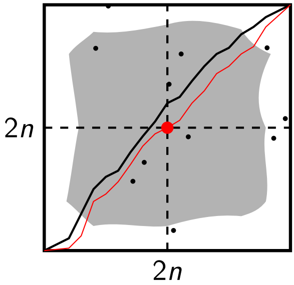

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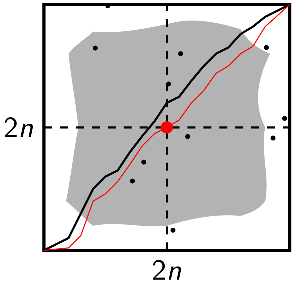

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Extend the definition of L_n from permutations to point processes

$$L(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \max\{k : \exists i_1 < \cdots < i_k \text{ s.t. } h(\mathbf{x}_{i_j}, \mathbf{x}_{i_\ell}) = 0\,,\ \forall j, \ell \leq k\}$$

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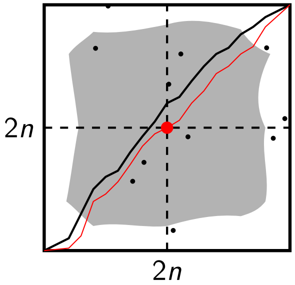

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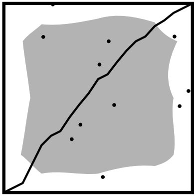
$$\mathbb{E}[L_{\text{Poisson}((x+y)^2)}] \geq \mathbb{E}[L_{\text{Poisson}(x^2)}] + \mathbb{E}[L_{\text{Poisson}(y^2)}]$$

$$\Rightarrow x^{-1}\mathbb{E}[L_{\text{Poisson}(x^2)}] \text{ converges by Fekete's theorem.}$$

Deuschel and Zeitouni's variational principle



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$$\exists C < \infty\,, \quad \frac{1}{C} \leq \frac{d\rho(\mathbf{x})}{d\mathbf{x}} \leq C\,,\ \forall \mathbf{x} \in [0,1]^2$$

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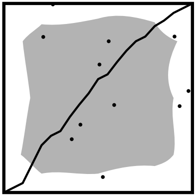

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$$\forall \epsilon > 0\,, \quad \lim_{n\to\infty} \rho^n(|n^{-1/2}L(\mathbf{x}_1,\dots,\mathbf{x}_n) - I(\rho)| > \epsilon) = 0\,,$$

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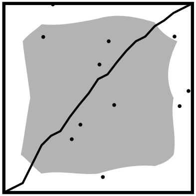

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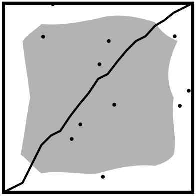

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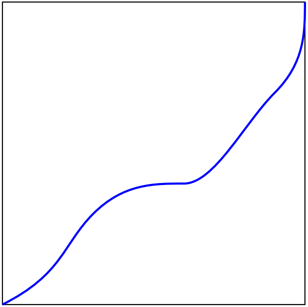
$$I(\rho,\gamma) = 2\int_0^1 \left[\frac{d\rho}{d\mathbf{x}}(\gamma(t))\,\frac{d\gamma^1}{dt}\cdot\frac{d\gamma^2}{dt}\right]^{1/2} dt\,.$$

Idea of proof and extension to Mallows



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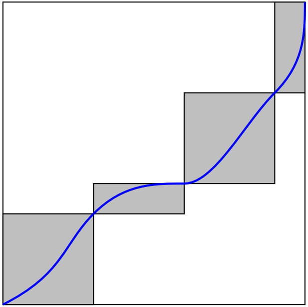

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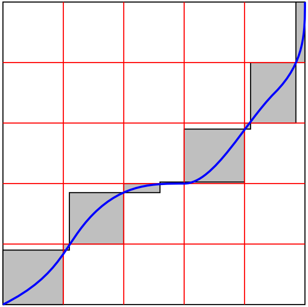

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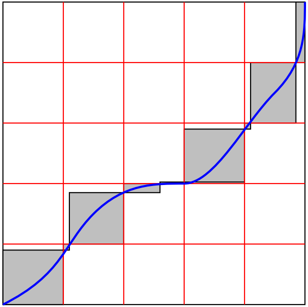

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Thm. (Mueller and S) Let $q_n(\beta) = \exp(-\beta/n)$,

$$\forall\epsilon > 0\,, \quad \lim_{n\to\infty} \mu_{n,q_n(\beta)}\{|n^{-1/2}L_n(\pi) - \mathcal{L}(\beta)| > \epsilon\} = 0\,,$$

where

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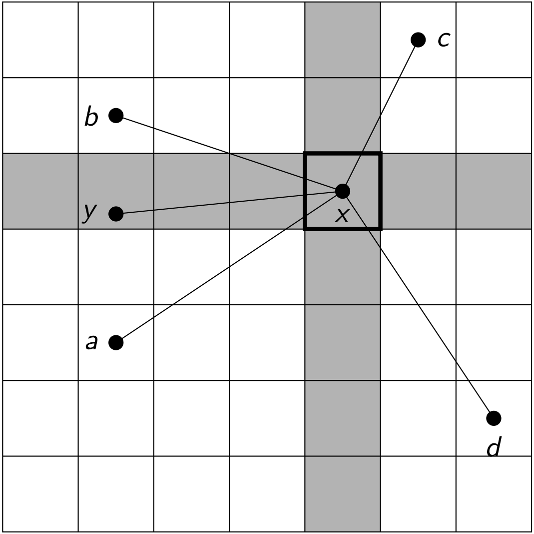
Weak conditional correlations



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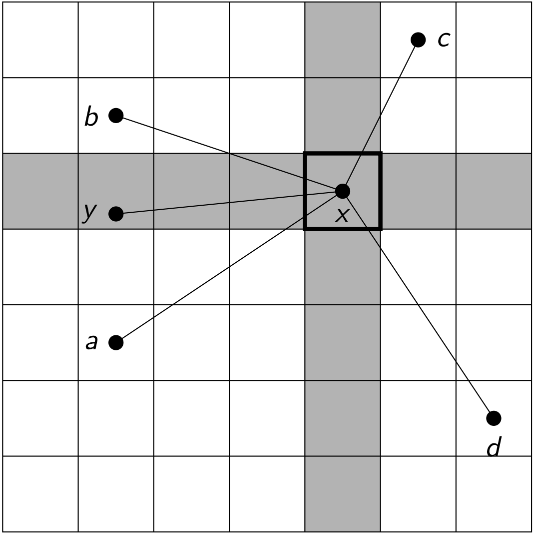

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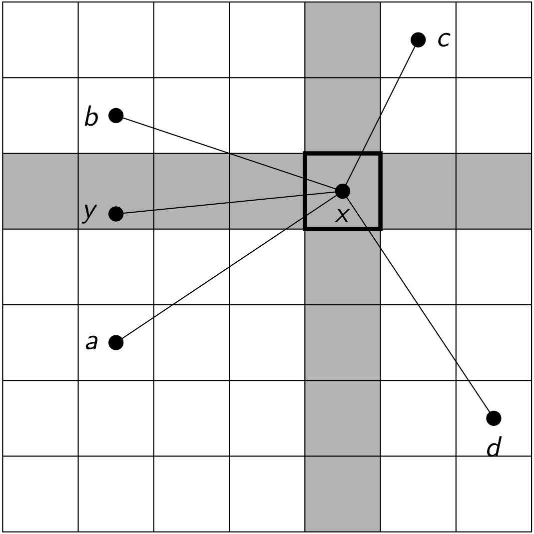

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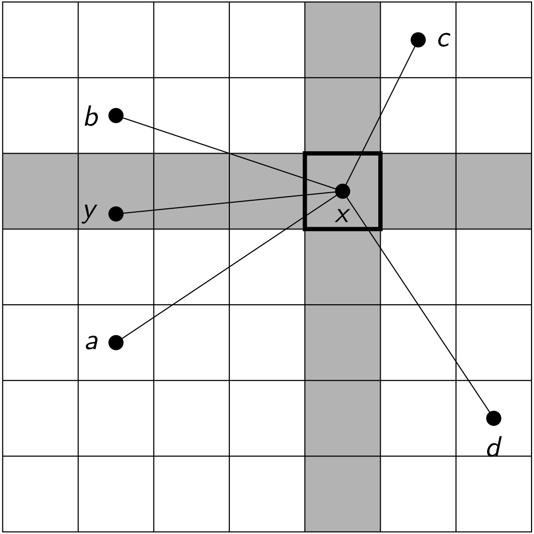

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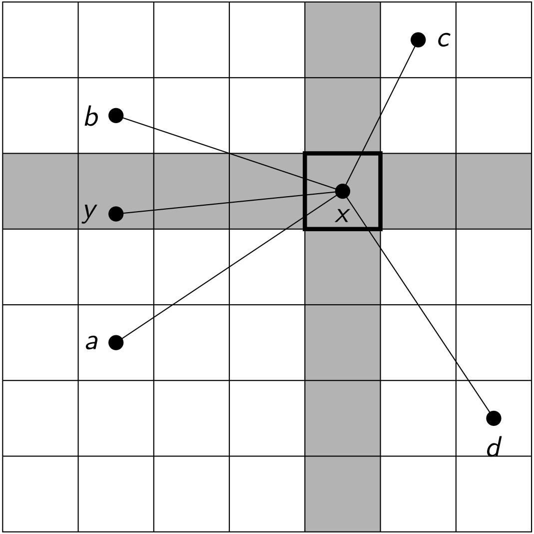

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H_n has $\frac{1}{n}$ factor.

Exponential interaction for box $O(1/k)$.

Coupling

"Coupling" two random variables X and Y: finding a common probability space $(\Omega, \mathcal{F}, P)$, joint distribution.

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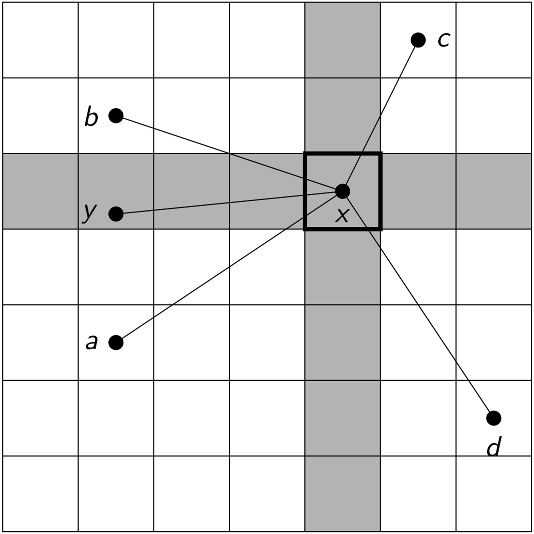
Weak conditional correlations redux



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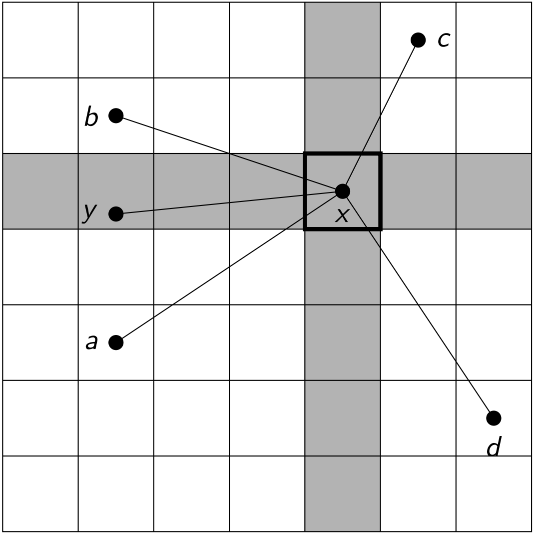

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So fraction of points that are not coupled to IID: $O(1/k)$.

Conclusion of proof

- For the empirical measure $\hat{\rho}_{n,\pi} = \frac{1}{n}\sum_{i=1}^{n} \delta_{(i/n,\pi_i/n)}$, $\mu_{n,q_n(\beta)}\{|\int_{[0,1]^2}\varphi\, d\hat{\rho}_{n,\pi} - \int_{[0,1]^2}\varphi\, d\rho_\beta| > \epsilon\} \to 0$ for each continuous φ and each $\epsilon > 0$.

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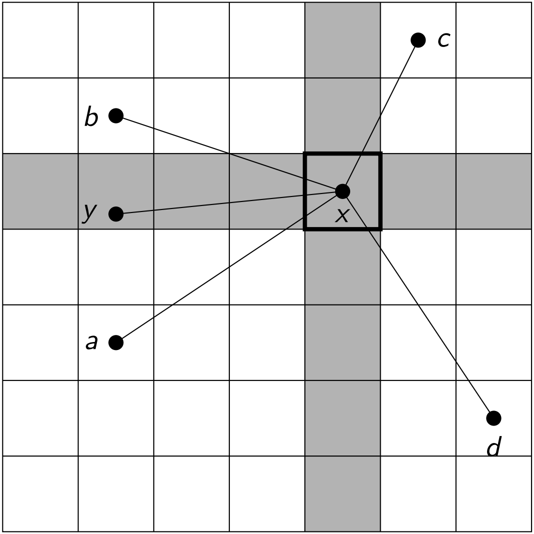
After a reparametrization $\rho_\beta(x'(x), y'(y)) \propto (1 - \beta xy)^{-2}$.

Bounds on the fluctuations



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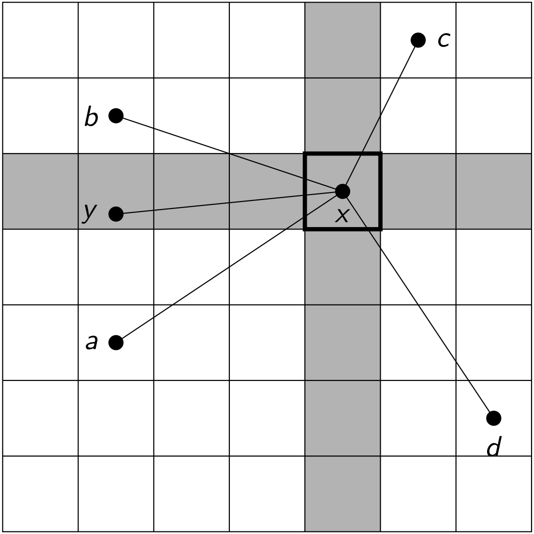

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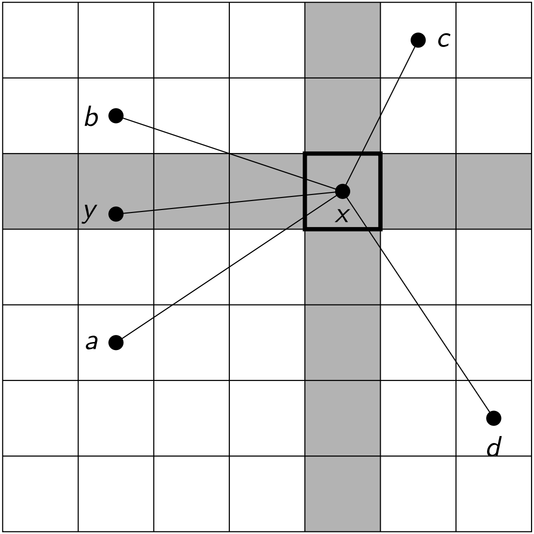

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$k = O(n^{1/4})$

14/18. Bounds on the counts

Four-square problem:

n_{21}	n_{22}
n_{11}	n_{12}

$$\mathbb{P}_q \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} = \mathbb{P}_1 \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} \cdot W_q \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix}$$

$$q^{n_{12}n_{21}} \frac{\{n_{11}+n_{12}\}!\{n_{11}+n_{21}\}!\{n_{12}+n_{22}\}!\{n_{21}+n_{22}\}!}{\{n_{11}\}!\{n_{12}\}!\{n_{21}\}!\{n_{22}\}!\{n_{11}+n_{12}+n_{21}+n_{22}\}!}$$

where $\{n\}! = [n]!/n!$.

Large deviations for 4-square

Stirling formula $\to$ relative entropy:

$$\frac{1}{n}\ln \mathbb{P}_1 \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} = \frac{1}{n}\ln\left(\frac{n!}{\prod_{i,j=1}^{2} n_{ij}!}\right) \to -\sum_{i,j=1}^{2} \rho_{ij}\ln\left(\frac{\rho_{ij}}{|\Lambda_{ij}|}\right)$$

for $n \to \infty$, with $n_{ij}/n \to \rho_{ij}$, where $|\Lambda_{ij}| =$ area of sub-square Λ_{ij}.

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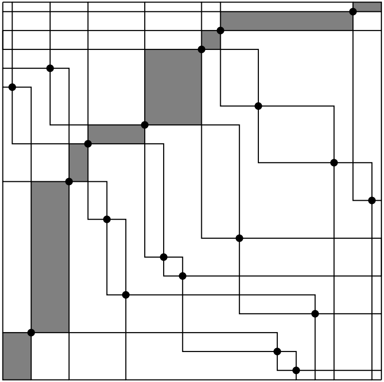
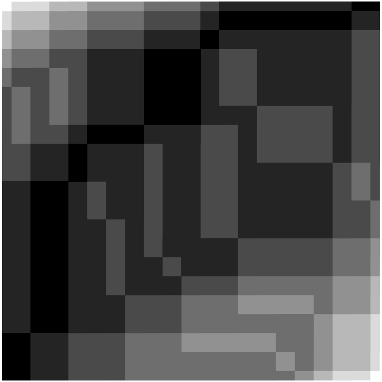
We can either settle for $O(n^{(3/8)+})$ bounds,

or we can prove $O(n^{(1/4)+})$ bounds along subsequences.

Cavity step?

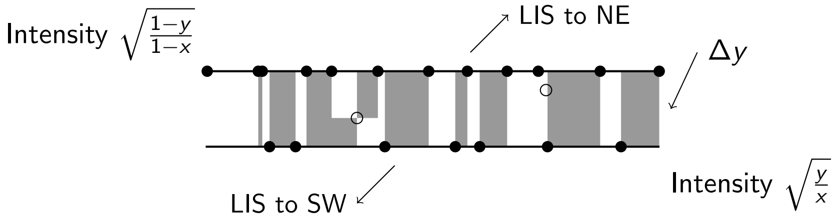




All we need to do is show that the area on the right hand picture is $O(n^{-1/2})$: each "box" is $O(n^{-1})$ and there are $O(n^{1/2})$ "boxes."

Local picture?

Aldous and Diaconis proved that on a horizontal slice, the length of the LIS behaves locally like a Poisson point process:



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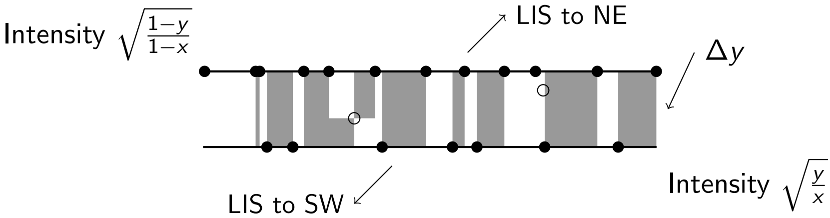

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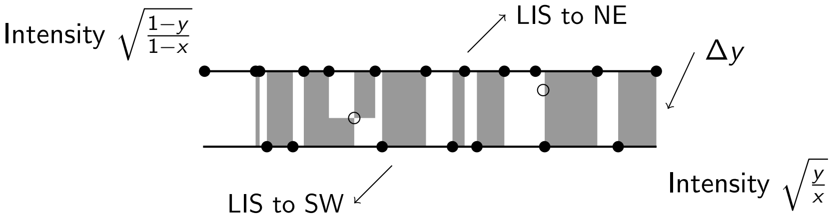

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Thanks for your attention!