Generalized Local Induction, Hasimoto's Map and Admissible Vortex Geometries

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Outline

A Few Words Before We Begin

If they understand the problem then you have done something.

Binormal Flow of a Vortex Filament

Vorticity $\omega = \nabla \times \mathbf{v}$ where $\omega, \mathbf{v} : \mathbb{R}^{3+1} \to \mathbb{R}^3$

Vortex Filament Plane curve localization, $\omega \neq 0$ on $\boldsymbol{\xi} : \mathbb{R} \to \mathbb{R}^3$

Self-Induced Motion Local evolution $\partial_t \xi = \kappa \left[\mathbf{T} \times \mathbf{T}' / |\mathbf{T}'| \right] = \kappa \mathbf{B}$

Binormal Flow and nonlinear PDE

Serret-Frenet $\mathbf{Y}_s' = \mathbf{A}(\kappa_s, \tau_s)\mathbf{Y}_s$ where $\mathbf{Y}_s = [\mathbf{T}\ \mathbf{N}\ \mathbf{B}]^T$

A Wave Function $\psi = \kappa e^{i\Phi}$ where $\Phi = \int^s \tau ds'$

Hasimoto's NLS $\mathbf{M} = (\mathbf{N} + i\mathbf{B})e^{i\Phi} \to i\partial_t \psi + \partial_{ss}\psi + \frac{1}{2}|\psi|^2\psi = 0$

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Hydrodynamic Problems

Vortices: A strange love hate relationship

A Millennium Problem

A fundamental problem in analysis is to decide whether smooth, physically reasonable solutions exist for the Navier–Stokes equations.

Role of Vorticity

Vorticity acts as a defining source of fluid flow in the absence of boundary influence, a kind of internal 'skeleton' that determines the structure of the flow.

Finite time vortex blowup and energy cascade

Fluid turbulence is observed to contain small domains of very large vorticity.

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Rescued by the Superfluid

...or how learned to stop worrying and love the quanta.

Landau's two-fluid model (1941)

For temperatures $0 < T < 2.172K = T_\lambda$, ^{4}He acts as if it were a mixture of two fluids,

$$\mathbf{v} = \mathbf{v}_n + \mathbf{v}_s, \qquad (1)$$

where $\mathbf{v}_n$ is a Navier-Stokes like fluid and $\mathbf{v}_s$ is is the superfluid component.

Lambda point transition

Superfluid Liquid Helium

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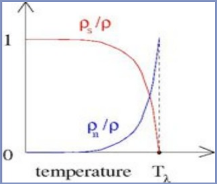

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- **Superfluid** No viscosity or entropy
- **Normal Fluid** Carries viscosity and entropy

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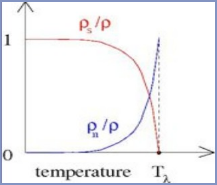

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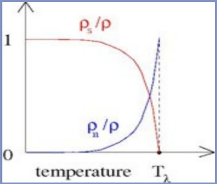

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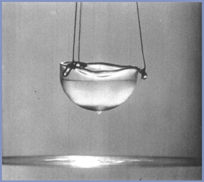

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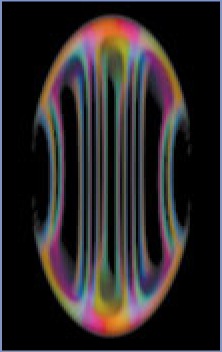
Quantization of vortex core size

R. Feynman (1955) Geometrically these nodal points must essentially form lines through the fluid. They are quantized vortex lines.

R. Donnelly (2001) The rotating bucket contains a uniform array of N vortices with a 'vortex-free strip' on the outside having quantized circulation.

Simulated vortices in a Bose-Einstein condensate



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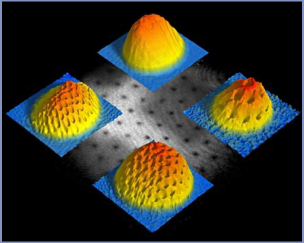
Gustavson vortex lattice



Turbulent Tangles

As good as it gets.

Theory and experiment

W. F. Vinen, J. J. Niemela (2002) Due to the restrictive nature of the quantized vortices, quantum turbulence must involve irregular arrangements that are called vortex tangles.

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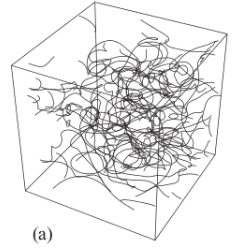
Numerical tangle, Tsubota, Araki, Nemirovskii (2000)



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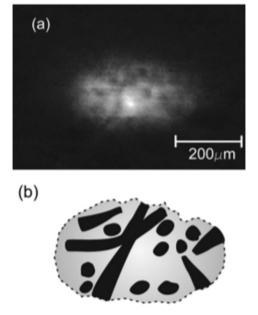

Fundamental Question: Redux

If vorticity is concentrated to lines then is it possible to predict its self-evolution?

Problem Statement

Hopes and Dreams

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Cottet and Soler (1988)

Beale and Majda (1982)

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Avoid Navier-Stokes evolution by:

- Constructing $\mathbf{v}$ from ω.
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Biot-Savart Integral

Recovering the velocity field from the vorticity field.

A modicum of hope

Helmholtz-Hodge $\mathbf{v} = \mathbf{v}_c + \mathbf{v}_d + \nabla\phi,$

Road map and hurdles

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Helmholtz-Hodge $\mathbf{v} = \mathbf{v}_c + \mathbf{v}_d + \nabla\phi$, $\mathbf{v}_c$: Irrotational

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Helmholtz-Hodge $\mathbf{v} = \mathbf{v}_c + \mathbf{v}_d + \nabla\phi$, $\mathbf{v}_d$: Incompressible

Biot-Savart Smooth, decaying , incompressible fields can be recovered via the Biot-Savart integral,

$$\mathbf{v}(\mathbf{x}) = \int_{\mathbb{R}^3} \frac{(\mathbf{x} - \omega) \times d\omega}{|\mathbf{x} - \omega|^3} \tag{2}$$

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Idea: Specify ω at $t = 0$ and using (2) recover $\mathbf{v}$ close to ω. Problem: Integrals can be analytically hard and numerically expensive. Solution: Constrain ω's geometry.

Constraining the Vorticity

Better living through geometry

Vortex filament

Let $\omega = \begin{cases} 1, & \mathbf{x} \in \xi \\ 0, & \mathbf{x} \notin \xi \end{cases}$,

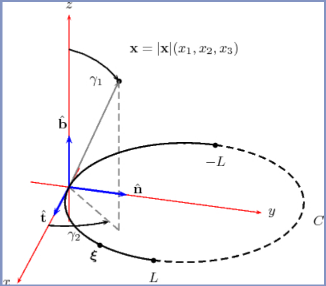
be such that

$\xi = (R\sin(\theta), R - R\cos(\theta), 0)$

where $R = \kappa^{-1} \in \mathbb{R}^+$ and

$\theta \in D_L = (-L, L]$, $L \in [0, \pi]$.

Vortex filament in $\mathbb{R}^3$



Legend

Black (solid)=Vortex, Black (Dashed+Solid)=circle parameterization, Blue=Local Coordinates (Serret-Frenet), Gray=Spherical Field Point, Red=Global Coordinates

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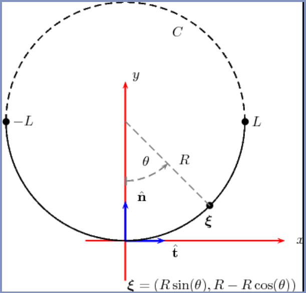

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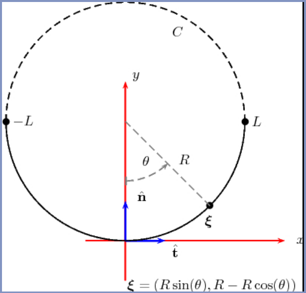

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What we hope to achieve

Prescription: $\boldsymbol{\xi} = (R\sin(\theta), R - R\cos(\theta), 0)$

For the previous arc, the Biot-Savart integral becomes

$$\mathbf{v}(\mathbf{x}) = \int_{D_L} \frac{(\mathbf{x} - \boldsymbol{\xi}) \times d\boldsymbol{\xi}}{|\mathbf{x} - \boldsymbol{\xi}|^3} \tag{3}$$

with elements

$$v_i(\mathbf{x}) = \int_{D_L} \frac{\epsilon_{ijk}(|\mathbf{x}|x_j - \xi_j)d\xi_k}{\left[|\mathbf{x}|^2 + |\boldsymbol{\xi}|^2 - 2|\mathbf{x}|(x_1\xi_1 + x_2\xi_2 + x_3\xi_3)\right]^{3/2}} \tag{4}$$

Goal

Study (3) for $\varepsilon = |\mathbf{x}|/R = |\mathbf{x}|\kappa \ll 1$ and hopefully find an asymptotic representation as $\varepsilon \to 0$.

Local Induction Approximation

Local in space and 'periodic in time'

Local Induction

A curved vortex line induces the local flow.

$$\mathbf{V}(\mathbf{x}) \approx \frac{\kappa}{4\pi} \ln\left(\frac{L}{|\mathbf{x}|}\right) \hat{\mathbf{b}}$$

Derivation Method

♣: Differential Equations/Geometry
♣: Matched Asymptotics
♣: Vortex Curve

Chronology

- 1906 Levi-Civita & Da Rios
- 1961 Arms & Hama
- 1965 R. Betchov
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Generalized Induction/Filament Equation

To Appear in the Journal of Mathematical Physics: arXiv:1102.2258

Theorem 1: GIE, directional component

$$\mathbf{V}_1(\varepsilon) = \varepsilon\beta_1\hat{\mathbf{t}} - \varepsilon\beta_2\hat{\mathbf{n}} + (\varepsilon\beta_2 + \varepsilon\beta_3 + \beta_4)\hat{\mathbf{b}}$$

The terms above imply movement in tangential direction , circulation about vortex core and binormal flow .

Theorem 2: Generalized Local Induction Equation

$$\mathbf{v}_\varepsilon(\mathbf{x}) = \kappa\left[\frac{72x_2^2F_1(\lambda,k)}{2} - \frac{8x_2^2E(L,k)}{(1-k^2)k}\right]\hat{\mathbf{b}}$$

The above formula implies that if $\boldsymbol{\xi}$ is the vortex arc then we have the standard curvature driven binormal flow $\partial_t\boldsymbol{\xi} \propto \kappa\hat{\mathbf{b}}$.

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A moving coordinate system for particle trajectories

Space Curve: $\boldsymbol{\xi} : \mathbb{R} \to \mathbb{R}^3$

- Tangent: $\mathbf{T} = \boldsymbol{\xi}'(s)$
- Normal: $\mathbf{N} = \dfrac{\mathbf{T}'}{||\mathbf{T}'||}$
- Binormal: $\mathbf{B} = \mathbf{T} \times \mathbf{N}$
- $(\mathbf{T}, \mathbf{N}, \mathbf{B})$ form an orthonormal set

Serret-Frenet Frame

Scalar Quantities

- Curvature: $\mathbf{T}' = \kappa\mathbf{N}$ gives $\kappa = ||\mathbf{T}'||$
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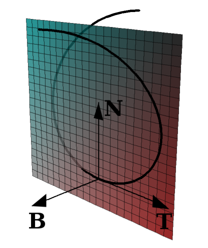

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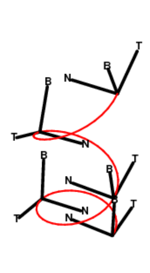

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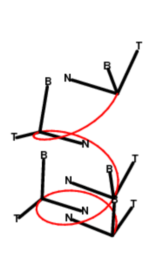

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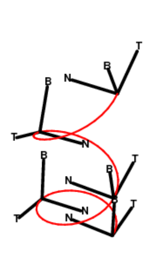

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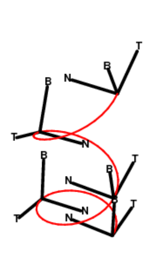

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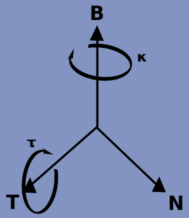

Serret-Frenet Equations

Recovering trajectories through kinematic quantities

Recall: $\mathbf{T}' = \kappa \mathbf{N}$, $\mathbf{B}' = -\tau \mathbf{N}$

- $|\mathbf{N}| = 1 \implies \langle \mathbf{N}', \mathbf{N} \rangle = 0$
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Key Point

Up to rotations, translations and calculus, κ, τ can be used to recover the regular curve ξ.

Notes

- SF are 9×9 system of ODEs
- SF are not generally autonomous
- The spectral problem is difficult

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$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}$$

Key Point

Up to rotations, translations and calculus, κ, τ can be used to recover the regular curve ξ.

Notes

- SF are 9×9 system of ODEs
- SF are not generally autonomous
- The spectral problem is difficult

Serret-Frenet Equations

Recovering trajectories through kinematic quantities

Recall: $\mathbf{T}' = \kappa \mathbf{N}$, $\mathbf{B}' = -\tau \mathbf{N}$

- $|\mathbf{N}| = 1 \implies \langle \mathbf{N}', \mathbf{N} \rangle = 0$
- $\mathbf{N}' = \langle \mathbf{N}', \mathbf{T} \rangle \, \mathbf{T} + \langle \mathbf{N}', \mathbf{B} \rangle \, \mathbf{B}$
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A clever transformation of binormal flow

Ingredients

- $\xi_t = \kappa \mathbf{B}$
- Conservation of Arclength
- Serret-Frenet Frame
- $\psi = \kappa e^{i\Phi}$, $\Phi = \int^s \tau \, ds'$
- $\mathbf{M} = (\mathbf{N} + i\mathbf{B})e^{i\Phi}$

Procedure

1. ξ_{st}: from binormal flow
2. ξ_{ts}: from $(\mathbf{T}, \mathbf{M}, \bar{\mathbf{M}})$
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The $SU(2)$ to $SO(3)$ map

Assuming a Darboux frame

Basis for $SU(2)$

$$\mathbf{U}_1 = -i\boldsymbol{\sigma}_z = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$$

$$\mathbf{U}_2 = i\boldsymbol{\sigma}_y = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

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The isometry $SO(3) \to SU(2)$

- Mapping 1: $(\mathbb{R}^3, \langle \cdot, \cdot \rangle) \to (SU(2), \cdot)$
- Mapping 2: $\mathbf{x} \in \mathbb{R}^3 \to \sum_{i=1}^{3} x_i U_i$
- Scalar Product: $\mathbf{A} \cdot \mathbf{B} = -2^{-1}\mathrm{tr}(\mathbf{AB})$

Image of $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ under $SU(2)$

There exists an $\Omega \in SU(2)$ such that $E_i = \Omega^{-1} U_i \Omega$ where (E_1, E_2, E_3) is the image of the natural frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$.

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Natural Serret-Frenet in the $SU(2)$

Fruits of the labor

Darboux Equations: $E_i = \Omega^{-1} U_i \Omega$

$$[U_1, \omega] = \kappa \cos(\Phi) U_2 + \kappa \sin(\Phi) U_3$$
$$[U_2, \omega] = -\kappa \cos(\Phi) U_1$$
$$[U_3, \omega] = -\kappa \sin(\Phi) U_1$$

Results: $\Omega' = \omega \Omega$

- $\omega = -\kappa \cos(\Phi) U_3 + \kappa \sin(\Phi) U_2$
- $\omega = \frac{i}{2} \begin{bmatrix} 0 & q \\ \bar{q} & 0 \end{bmatrix}$
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Key Point: A procedure for curve reconstruction

We now have the inversion of the Hasimoto transformation:

$$VFE \to \text{NLS} \to \kappa, \tau \to q \to \omega \to \Omega \to E_1 \to \mathbf{e}_1 \to \boldsymbol{\xi}$$

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Fruits of the labor

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Conclusions

Thank you for your attention

Key Points

1. A vortex filament with nontrivial curvature will always display a binormal flow
2. Binormal Flow maps onto NLS
3. Using NLS one can theoretically recover the vortex geometry
4. Serret-Frenet obstructs analytic results but can be circumnavigated by using the $SU(2)$ formalism

Future Work

- What PDE are possible for the case of vortex tubes?

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