Existence of Thermodynamic Limit for Interacting Quantum Particles in Random Media

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Outline

1. Model
   - Multiparticle Random Schrödinger Operator
   - Thermodynamic Limit and Problem We Study
2. Existence Theorems
   - Main Result: Energy Convergence
   - Convexity and Entropy Convergence
3. Further Advances: Poisson Pieces Model

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$$\mathfrak{H}^n = \begin{cases} \bigotimes^n \mathfrak{H} = L^2(\mathbb{R}^{nd}), & \text{for classical particles,} \\ \bigwedge^n \mathfrak{H} = L^2_-(\mathbb{R}^{nd}), & \text{for fermions,} \\ \mathrm{Sym}^n \mathfrak{H} = L^2_+(\mathbb{R}^{nd}), & \text{for bosons} \end{cases}$$

Thermodynamic Limit and Problem We Study

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Example

Let $E_\omega(\Lambda, n)$ be the ground state energy of $H_\omega(\Lambda, n)$. Question:

$$E_\omega(\Lambda, n) \xrightarrow[\Lambda \to \mathbb{R}^d, n/|\Lambda| \to \rho]{} ?$$

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- $E_\omega(\Lambda, n, 0)$ is the ground state energy of $H_\omega(\Lambda, n)$

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Then the energy admits thermodynamic limit:

$$\frac{E_\omega(\Lambda, n, S)}{n} \xrightarrow{L^2_\omega} \varepsilon(\rho, \sigma), \quad \Lambda \to \mathbb{R}^d, \frac{n}{|\Lambda|} \to \rho, \frac{S}{n} \to \sigma \geqslant 0.$$

The energy density $\varepsilon(\rho, \sigma)$ *is a deterministic function.*

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- Entropy as a function of energy admits thermodynamic limit:

$$\frac{S_\omega(E, \Lambda, n)}{n} \to \sigma(\rho, \varepsilon), \quad \Lambda \to \mathbb{R}^d, \ \frac{n}{|\Lambda|} \to \rho, \ \frac{E}{n} \to \varepsilon \in \operatorname{Ran} \varepsilon(\rho, \cdot),$$

where $\sigma(\rho, \cdot)$ is an inverse of $\varepsilon(\rho, \cdot)$.

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- Reference model for $d = 1$:
  1. domain Λ becomes $[0, L]$
  2. consider Poisson point process on $\mathbb{R}$
  3. points inside $[0, L]$ define positions of walls
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- Description of $\Psi_\omega(\Lambda, n)$ is obtained for small ρ. In particular:
  1. Loose functional subspace is found
  2. Slater determinant type structure
  3. Autocorrelation function and decorrelation length are estimated

The End

Thank you for your attention!