Fractional moment localization in a system of interacting particles in an alloy-type random potential

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joint work with Simone Warzel

The model

Consider a random Schrödinger operator for a system of n interacting particles in $\mathbb{R}^d$, acting on $L^2(\mathbb{R}^d)^n \cong L^2(\mathbb{R}^{dn})$:

$$H^{(n)}(\omega) = \sum_{j=1}^{n}(-\Delta_j + V_0(x_j) + V(\omega, x_j)) + \alpha \sum_{j<k} W(x_j - x_k)$$

- $V_0 \in L^\infty(\mathbb{R}^d)$: $\mathbb{Z}^d$-periodic background potential
- $V(\omega)$: alloy-type random potential:

$$V(\omega, x) = \sum_{\zeta \in \mathbb{Z}^d} \eta_\zeta(\omega) U(x - \zeta)$$

- $U \in L_c^\infty(\mathbb{R}^d)$, $\sum_\zeta U(x - \zeta) \geq 1$ for all $x \in \mathbb{R}^d$
- $(\eta_\zeta)_{\zeta \in \mathbb{Z}^d}$: iid random variables with density $\rho \in L_c^\infty(\mathbb{R})$
- $W \in L^\infty(\mathbb{R}^d)$: exponentially decaying interaction potential, strength controlled via $\alpha \geq 0$

Goal:

Dynamical localization in an interval $I = [E_0^{(n)}, E_0^{(n)} + \eta^{(n)}]$ at the bottom $E_0^{(n)} = \inf \sigma(H^{(n)})$ of the spectrum:

$$\mathbb{E}\left[\sup_{t\in\mathbb{R}} \left\| \mathbf{1}_{B_1(\mathbf{x})} e^{-itH^{(n)}} P_I(H^{(n)}) \mathbf{1}_{B_1(\mathbf{y})} \right\| \right] \leq C e^{-\mu \, \mathrm{dist}_H(\mathbf{x},\mathbf{y})}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{dn}$.

- $P_I(H^{(n)})$ = spectral projection of $H^{(n)}$ onto I
- $\mathrm{dist}_H(\mathbf{x}, \mathbf{y}) = \max\{\max_j \min_k |x_j - y_k|, \max_j \min_k |y_j - x_k|\}$
  = Hausdorff distance of the sets $\{x_j \,|\, 1 \leq j \leq n\}$ and $\{y_j \,|\, 1 \leq j \leq n\}$

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Related results:

Aizenman/Warzel '09, Chulaevsky/Suhov '09, Chulaevsky/Boutet de Monvel/Suhov '11, ...

Fractional moment localization

Definition

A bounded interval I is a regime of fractional moment (FM) localization in I if and only if there exist $s \in (0,1)$ and $C, \mu > 0$ such that

$$\sup_{\substack{\Omega \subset \mathbb{R}^d \\ \text{open, bd.}}} \sup_{\substack{\operatorname{Re} z \in I \\ 0 < |\operatorname{Im} z| < 1}} \mathbb{E}\left[\| \mathbf{1}_{B_1(\mathbf{x})} (H_\Omega^{(n)} - z)^{-1} \mathbf{1}_{B_1(\mathbf{y})} \|^s \right] \leq C e^{-\mu \operatorname{dist}_H(\mathbf{x},\mathbf{y})}$$

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Lemma

FM localization in I implies dynamical localization in I.

Main result

Theorem

Assume that the one-particle operator exhibits FM localization in the interval $[E_0^{(1)}, E_0^{(1)} + \eta^{(1)}]$ (*cf. Aizenman et al. '06*).

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Current work:

Extension of these results to interactions with sufficiently fast polynomial decay