Quasiperiodic Schrödinger operators with Rough Potentials

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Quasiperiodic operators

Consider quasiperiodic Schrödinger operators on $\ell^2(\mathbb{Z})$ defined as

$$(\mathfrak{h}_\theta u)(n) = u(n+1) + u(n-1) + \lambda f(\theta + n\omega)u(n).$$

With the assumptions

- $f \in \mathcal{C}^\gamma(\mathbb{T}, \mathbb{R})$ for $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $\gamma > 0$.
- The frequency: $\omega \in \mathbb{T}$. The phase: $\theta \in \mathbb{T}$.

Note

- For $f(x) = 2\cos(2\pi x)$ this is the almost Mathieu operator.
- For irrational ω the spectrum is constant in θ.
- Let $S(\omega) = \cup_\theta \sigma(\omega, \theta)$.

The formal eigenvalue equation is

$$Eu(n) = u(n+1) + u(n-1) + \lambda f(\theta + n\omega)u(n).$$

We associate transfer matrices to the eigenvalue equation,

$$A^E(\theta) = \begin{pmatrix} E - \lambda f(\theta) & -1 \\ 1 & 0 \end{pmatrix} \in \mathcal{C}\left(\mathbb{T}, \mathbf{SL}_2(\mathbb{R})\right).$$

Composing these defines forward matrix cocyles

$$A_k^E(\theta) = A(\theta + (k-1)\omega, E) \cdots A(\theta + \omega, E)A(\theta, E). \quad k > 0;$$

$$A_k^E(\theta) = \left[A_{-k}(\theta - k\omega)\right]^{-1}, \quad k < 0; \qquad A_0 = I.$$

For $k \in \mathbb{Z}$,

$$\begin{pmatrix} u(k) \\ u(k-1) \end{pmatrix} = A_k^E(\theta) \begin{pmatrix} u(0) \\ u(-1) \end{pmatrix}.$$

- The cocycles are subadditive,

$$\ln \|A_{m+n}^E(\theta)\| \leq \ln \|A_m^E(\theta)\| + \ln \|A_n^E(T^m\theta)\|, \quad \frac{1}{n} \ln \|A_n^E(\theta)\| \to \mathcal{L}(A^E) a.s..$$

Irrational numbers, continued fractions

Any $\omega \in (0,1)$ can be represented as a continued fraction:

$$\omega = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

with $a_i \in \mathbb{Z}^+$. It is thus possible to encode ω as $[a_1, a_2, a_3, \cdots]$. The expansion terminates iff ω is rational. Moreover, the rational approximants of ω are the truncations, $\frac{p_n}{q_n} := [a_1, a_2, \cdots, a_n]$; with $(p_n, q_n) = 1$.

- ω is $\kappa-$Diophantine, $\kappa > 0$, if $q_{n+1} < q_n^{1+\kappa}$ for all large n.
- ω is of finite type if the sequence a_i is uniformly bounded.
- For all ω, approximants obey

$$\left|\omega - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}}$$

Spectral limit

Theorem (Jitomiskaya+M.)

Let a sampling function $f \in \mathcal{C}^\gamma(\mathbb{T})$ with $\gamma > \frac{1}{2}$ define a Schrödinger operator,

$$(\mathfrak{h}_\theta u)(n) = u(n+1) + u(n-1) + \lambda f(\theta + n\omega)u(n).$$

In the regime of almost everywhere positive Lyapunov exponent, for every irrational ω, there exists a sequence of rational approximants $\frac{p_n}{q_n} \to \omega$ so that,

$$|S\left(\frac{p_n}{q_n}\right)| \to |S(\omega)|.$$

Without assumptions on the Lyapunov exponent, for any $1 > \gamma > 0$, if ω is not $\frac{1-\gamma}{\gamma}$-Diophantine then there exists a sequence of rational approximants $\frac{p_n}{q_n} \to \omega$ so that,

$$|S\left(\frac{p_n}{q_n}\right)| \to |S(\omega)|.$$

Spectral continuity properties

- (Aubry, Andre 1980) Conjecture: The Lebesgue measure of the spectrum of the almost Mathieu operator for any irrational rotation is $4|1-|\lambda||$.
- (Avron, Mouche, Simon 1990) Calculate Lebesgue measure of the spectrum of the almost Mathieu operator for rational frequencies, and showed Hausdorff$-\frac{1}{2}$ continuity of spectral set in the frequency for $\mathcal{C}^1$ potentials.
- (Last 1994) Proves Aubry-Andre conjecture for irrational frequencies of non-finite type. Results hold up to $\mathcal{C}^1$.
- (Jitomirskaya, Krasovsky 2001) Proves almost lipshitz continuity of spectral sets in the Hausdorff metric for all Diophantine rotations in the regime of positive Lyapunov exponents for analytic potentials.

Uniform upper semicontinuity of the Lyapunov exponent

Lemma (Jitomirskaya+M.)

For any $\eta > 0$ and $A \in \mathcal{C}(X, \mathbf{GL}_{n\times n}(\mathbb{C}))$. (X, T) compact and uniquely ergodic. There is $k_\eta < \infty$, $\delta > 0$ so that, $k > k_\eta$ implies for $D \in \mathcal{C}(X, \mathbf{GL}_{n\times n}(\mathbb{C}))$ and

$$\|D - A\|_0 < \delta,$$

we have for all θ,

$$\|D_k(\theta)\| < e^{k(\mathcal{L}(A)+\eta)}.$$

- For $\delta = 0$ this is a theorem of Furman's.

Corollary

With assumptions as above,

$$\|A_k - D_k\|_0 \leq \delta e^{\{k(\mathcal{L}(A)+\eta)\}}$$

Matrix formulation

The eigenvalue equation can be written as a matrix restricted to an interval
$I = [x_1, x_2]$,

$$(H-E)_I(\theta) = \begin{bmatrix} f(\theta + x_1\omega) - E & 1 & 0 & \cdots & & \\ 1 & f(\theta + (x_1+1)\omega) - E & 1 & 0 & & \cdots \\ & & \ddots & & & \\ & & & \ddots & & \\ & \cdots & & 0 & 1 & f(\theta + x_2\omega) - E \end{bmatrix}$$

The Green's function is then just the matrix inverse, $G_I = (H - E)_I^{-1}$.
Moreover, the determinants are related to the transfer matrices,

$$A_k^E(\theta) = \begin{bmatrix} \det(H-E)_{[1,k]}(\theta) & -\det(H-E)_{[1,k-1]}(\theta+\omega) \\ \det(H-E)_{[1,k-1]}(\theta) & -\det(H-E)_{[1,k-2]}(\theta+\omega) \end{bmatrix}$$

The spectral sets are continuous in frequency in the Hausdorff metric.

Theorem (Jitomirskaya+M.)

Suppose f is $\gamma-$Hölder continuous, $1 \geq \gamma > 0$.

- *For $E \in S(\omega)$ and for small enough $|\omega - \omega'|$, there exists an $E' \in S(\omega')$ so that*

$$|E - E'| < C_f |\omega - \omega'|^{\frac{\gamma}{1+\gamma}}.$$

- *On the other hand, if $\mathcal{L}(A^E)$ is positive for almost every E, and ω is Diophantine. Let $\gamma > \beta > 0$. Given $\zeta > 0$ there is a B_ζ, $0 < |B_\zeta| < \zeta$ so that for $E \in S(\omega) \backslash B_\zeta$ and any rational approximant ω' near ω, there exists $E' \in S(\omega')$ such that*

$$|E - E'| < C_f |\omega - \omega'|^{\beta}.$$

The constant $C_f > 0$ does not depend on ω or ω'.

Demonstration

For positive Lyapunov exponent,

- For $E \in S(\omega)$, take nearby generalized eigenfunction at energy E_0. It has growth which is polynomially bounded.
- Cramer's formula implies,

$$|G^{E_0}_{[x_1,x_2]}(x_l, x_1)| \le \frac{|\det[(H - E_0)_{[x_l,x_1]}(T^{x_1}\theta)]|}{|\det[(H - E_0)_{[x_2,x_1]}(T^{x_1}\theta)]|}$$

- The numerator grows exponentially, the corollary of uniform uppersemicontinuity implies the denominator grows exponentially along some intervals.
- For $x = x_\ell, x_r$ let x_1, x_2 be properly chosen nearby sites,

$$\psi(x) = -G_{[x_1,x_2]}(x, x_1)\psi(x_1 - 1) - G_{[x_1,x_2]}(x, x_2)\psi(x_2 + 1)$$

- Let ψ' be the restriction of ψ to $[x_\ell, x_r]$ exponentially small $\|(\mathfrak{h}_\omega - E_0)\psi'\|$
- Approximate,

$$\left\|(E - \mathfrak{h}_{\omega',\theta'})\,\psi'\right\| \le |E - E_0|\|\psi'\| + \left\|(E_0 - \mathfrak{h}_{\omega,\theta})\psi'\right\| + \left\|(\mathfrak{h}_{\omega,\theta} - \mathfrak{h}_{\omega',\theta'})\,\psi'\right\|$$

- Finally, the last term is bounded by $|f(\theta + n\omega) - f(\theta' + n\omega')|$ for $n \in [x_\ell, x_r]$ (properly chosen θ'). Variational principle implies a nearby $E' \in S(\omega')$.

Demonstration

For positive Lyapunov exponent, $\gamma > \beta > 1/2$.

- $S(p/q)$ consists of q bands, with $q-1$ (possibly empty) gaps.
- We use the continuity of the gaps of the spectrum.
- Let $\omega_n = p_n/q_n$, so that $q_n|\omega - p_n/q_n|^\beta \to 0$.
- To show $|S(\omega)| \geq \limsup |S(\omega_n)|$, consider a large collection of gaps $G_\epsilon(\alpha)$ of $S(\omega)$; $|[\inf S(\omega), \sup S(\omega)] \backslash G_\epsilon(\omega)| < |S(\omega)| + \epsilon$.
  - $|G_\epsilon(\omega) \Delta G_\epsilon(p_n/q_n)| < 2q_n C_f|\omega - p_n/q_n|^\beta$.
  - Thus $|S(\omega)| + \epsilon \geq \limsup |S(p_n/q_n)|$.
- On the other hand, let $[a_{n,i}, b_{n,i}]$ for $1 \leq i \leq q_n$ be the q_n bands of the spectrum of $S(p_n/q_n)$

$$S(\omega) \subset \bigcup_{i=1}^{q_n} [a_{n,i} - C_f|\omega - p_n/q_n|^\beta, b_{n,i} + C_f|\omega - p_n/q_n|^\beta] \cup B_\zeta.$$

  - Thus $|S(\omega)| \leq |S(p_n/q_n)| + 2q_n C_f|\omega - p_n/q_n|^\beta + \zeta$.
  - Taking the limit of q_n shows $|S(\omega)| \leq \liminf |S(p_n/q_n)| + \zeta$.
- Let $\zeta, \epsilon \to 0$ to obtain $\lim |S(p_n/q_n)| = S(\omega)$ □

Thank you!