For the details, see [JP],[JOPP],[AJPP] and the references therein. We consider the following setting of nonequilibrium statistical mechanics. The system we consider is given by a C^*-algebra

$$\mathcal{O} := \otimes_k \mathcal{O}_k,$$

associated with free time evolution

$$\tau_0^t = \otimes_k \tau_k^t.$$

The initial state is

$$\omega = \otimes_k \omega_k,$$

where each ω_k is (β_k, τ_k)-KMS state. The dynamics τ is given by perturbing the free dynamics by an interaction $V \in \mathcal{O}$. If there exists a state ω_+ such that

$$\omega_+(A) = \lim_{t\to\infty} \omega_t(A), \quad A \in \mathcal{O},$$

we call it nonequilibrium steady state. We furthermore assume the existence of time reversal symmetry Θ, i.e., an antilinear automorphism such that

$$\tau_t \circ \Theta = \Theta \circ \tau_{-t}, \quad \Theta^2 = id.$$

For a state ϕ, its time reversal $\bar{\phi}$ is given by $\bar{\phi}(a) = \phi(\Theta(a^*))$. We consider the situation that $\bar{\omega} = \omega$. We have $\bar{\omega}_t = \omega_{-t}$, so ω_t is not time reversal invariant anymore, in general.

We want to estimate *how far* ω_t and ω_{-t} are. To do that, we consider measures like

$$D(\omega_1, \omega_2) := 1 - \frac{1}{2} \left\| \omega_1 - \omega_2 \right\|,$$

$$\inf_{0 \le T \le 1} \{ \omega_1(T) \;:\; \omega_2(T) \ge 1 - \varepsilon, \}, \quad 0 < \varepsilon < 1.$$

For $s \in \mathbb{R}$, we define

$$E_s(\omega_t, \omega_{-t}) = \log(\Delta^s_{\omega_t, \omega_{-t}} \xi_{\omega_{-t}}, \xi_{\omega_{-t}}).$$

Here ξ_ϕ is the representative of ϕ in the natural positive cone. The main result in [JOPS] is the following.

Theorem 0.1 *[JOPS] Assume that the limit*

$$e(s) = \lim_{t\to\infty} \frac{1}{2t} E_s(\omega_t, \omega_{-t})$$

exists and differentiable in some open interval I which includes $[0,1]$, and $e'(1) > 0$. Then

1.

$$\lim_{t\to\infty}\frac{1}{2t}\log D(\omega_t.\omega_{-t}) = e(\frac{1}{2}).$$

2.

$$\underline{B}(r) = \bar{B}(r) = B(r) = -\sup_{s\in[0,1[}\frac{-sr-e(s)}{1-s}.$$

for

$$\bar{B}(r) = \inf_{\{T_t\}}\left\{\limsup_{t\to\infty}\frac{1}{2t}\log\omega_t(1-T_t)\;\middle|\;\limsup_{t\to\infty}\frac{1}{2t}\log\omega_{(-t)}(T_t) < -r\right\},$$

$$\underline{B}(r) = \inf_{\{T_t\}}\left\{\liminf_{t\to\infty}\frac{1}{2t}\log\omega_t(1-T_t)\;\middle|\;\limsup_{t\to\infty}\frac{1}{2t}\log\omega_{(-t)}(T_t) < -r\right\},$$

$$B(r) = \inf_{\{T_t\}}\left\{\lim_{t\to\infty}\frac{1}{2t}\log\omega_t(1-T_t)\;\middle|\;\limsup_{t\to\infty}\frac{1}{2t}\log\omega_{(-t)}(T_t) < -r\right\}.$$

where in the last case the infimum is taken over all families of tests $\{T_t\}$ *for which*

$$\lim_{t\to\infty}\frac{1}{2t}\log\omega_t(1-T_t)$$

exists.

3. For $\epsilon\in]0,1[$,

$$\bar{B}_\epsilon = \underline{B}_\epsilon = B_\epsilon = -e'(1).$$

where

$$\bar{B}_\epsilon = \inf_{\{T_t\}}\left\{\limsup_{t\to\infty}\frac{1}{2t}\log\omega_t(1-T_t)\;\middle|\;\omega_{(-t)}(T_t)\le\epsilon\right\},$$

$$\underline{B}_\epsilon = \inf_{\{T_t\}}\left\{\liminf_{t\to\infty}\frac{1}{2t}\log\omega_t(1-T_t)\;\middle|\;\omega_{(-t)}(T_t)\le\epsilon\right\},$$

$$B_\epsilon = \inf_{\{T_t\}}\left\{\lim_{t\to\infty}\frac{1}{2t}\log\omega_t(1-T_t)\;\middle|\;\omega_{(-t)}(T_t)\le\epsilon\right\},$$

The rate $e'(1)$ is called entropy production.

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