**MATH 129-020:**
**SIMS**
**TEST 3**

SPRING 2019

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| Name | Key |
| I.D. Number | |

| Question | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| 6 | 10 | |
| Total | 60 | |

(1) a) Write the following as a finite geometric sum and find its value.

$$\frac{3}{4}+\frac{3}{8}+\frac{3}{16}+\frac{3}{32}+\cdots+\frac{3}{2^{14}}$$

$$= \sum_{m=2}^{14} 3\left(\tfrac{1}{2}\right)^m = 3\left(\tfrac{1}{2}\right)^2 \cdot \sum_{m=0}^{12} \left(\tfrac{1}{2}\right)^m$$

$$= \frac{3}{4}\cdot\left(\frac{1-\left(\frac{1}{2}\right)^{13}}{1-\frac{1}{2}}\right)$$

$$= \frac{3}{2}\left(1-\frac{1}{2^{13}}\right)$$

Also:

$$\sum_{m=2}^{14} 3\left(\tfrac{1}{2}\right)^m$$

$$= \frac{3\left(1-\left(\frac{1}{2}\right)^{15}\right)}{1-\frac{1}{2}}$$

$$-3-\frac{3}{2}$$

b) Calculate

$$\sum_{n=2}^{\infty}\frac{3^{2n}+6(-4)^n}{5^{3n}}$$

$$\sum_{n=2}^{\infty}\frac{3^{2n}+6(-4)^n}{5^{3n}} = \sum_{n=2}^{\infty}\frac{3^{2n}}{5^{3n}} + \sum_{n=2}^{\infty}\frac{6(-4)^n}{5^{3n}}$$

$$= \sum_{n=2}^{\infty}\left(\frac{3^2}{5^3}\right)^n + 6\sum_{n=2}^{\infty}\left(\frac{-4}{5^3}\right)^n$$

$$= \left(\frac{9}{125}\right)^2 \cdot \frac{1}{1-\frac{9}{125}} + 6\left(\frac{-4}{125}\right)^2 \cdot \frac{1}{1-\left(\frac{-4}{125}\right)}$$

$$\simeq 0.01154$$

(2) Determine whether the following series converge or diverge. Write a sentence describing the convergence test you used and state your conclusion. For full/partial credit, show all work necessary to reach your conclusions.

a)

$$\sum_{n=3}^{\infty} \frac{5n^2+4}{7n^5+2n^3}$$

$$a_n = \frac{5n^2+4}{7n^5+2n^3} = \frac{n^2(5+\frac{4}{n^2})}{n^5(7+\frac{2}{n^2})} \approx \frac{5}{7n^3}$$

Let $b_n = \frac{5}{7n^3}$. By p-test, $\sum_{n=3}^{\infty} b_n$ converges since $p=3>1$.

By limit comparison: $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2(5+\frac{4}{n^2})}{n^5(7+\frac{2}{n^2})} \cdot \frac{7n^3}{5} = 1 > 0$

Shows that $\sum_{n=3}^{\infty} a_n$ converges as well.

b)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+3}}$$

Use Alternating Series test.

i) $a_n = \frac{1}{\sqrt{n^2+3}} > 0$ for all $n \geq 1$.

ii) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n^2+3}} = 0$

iii) $n < n+1 \Rightarrow n^2 < (n+1)^2 \Rightarrow n^2+3 < (n+1)^2+3$

$$\Rightarrow \frac{1}{(n+1)^2+3} < \frac{1}{n^2+3}$$

$$\Rightarrow a_{n+1} = \sqrt{\frac{1}{(n+1)^2+3}} < \sqrt{\frac{1}{n^2+3}} = a_n$$

This series then converges by the Alternating Series Test.

(3) Consider the following power series.

$$\sum_{n=0}^{\infty} \frac{(-1)^n n^2}{3^{2n}}(x+2)^n$$

a) Find the radius of convergence.

Let $a_n = \dfrac{(-1)^n n^2 (x+2)^n}{3^{2n}}$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)^2 |x+2|^{n+1}}{3^{2(n+1)}} \cdot \frac{3^{2n}}{n^2 |x+2|^n}$$

$$= \frac{|x+2|}{3^2} \quad \Rightarrow \quad R = 3^2 = 9.$$

b) Find the interval of convergence. You do not need to consider the endpoints.

The center is $a = -2$.

In this case, the interval of convergence is

$$(a-R, a+R) = (-2-9, -2+9)$$

$$= (-11, 7)$$

(4) Find the Taylor polynomial of degree 4, i.e. $P_4(x)$, for the function $f(x) = \cos(x)$ centered at $a = \pi/2$.

$f(x) = \cos(x)$ | $f(\frac{\pi}{2}) = 0$

$f'(x) = -\sin(x)$ | $f'(\frac{\pi}{2}) = -1$

$f''(x) = -\cos(x)$ | $f''(\frac{\pi}{2}) = 0$

$f'''(x) = \sin(x)$ | $f'''(\frac{\pi}{2}) = 1$

$f^{(4)}(x) = \cos(x)$ | $f^{(4)}(\frac{\pi}{2}) = 0$

$$P_4(x) = \sum_{k=0}^{4} \frac{f^{(k)}(\frac{\pi}{2})}{k!}\left(x - \frac{\pi}{2}\right)^k$$

$$= f(\tfrac{\pi}{2}) + f'(\tfrac{\pi}{2})\left(x - \tfrac{\pi}{2}\right) + \frac{f''(\frac{\pi}{2})}{2!}(x - \pi/2)^2 + \frac{f'''(\frac{\pi}{2})}{3!}\left(x - \tfrac{\pi}{2}\right)^3 + \frac{f^{(4)}(\frac{\pi}{2})}{4!}\left(x - \tfrac{\pi}{2}\right)^4$$

$$= 0 - \left(x - \tfrac{\pi}{2}\right) + 0 + \frac{1}{3!}\left(x - \tfrac{\pi}{2}\right)^3 + 0$$

$$= -\left(x - \tfrac{\pi}{2}\right) + \frac{1}{6}\left(x - \tfrac{\pi}{2}\right)^3$$

(5) Consider the function

$$f(x) = \frac{1}{1-2x}.$$

a) Find the Taylor series centered at $a = 0$ for this function.

Two ways to work this:

1) $f$ looks like geometric series:

$$f(x) = \frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n \cdot x^n.$$

2) Take derivatives:

$$f(x) = (1-2x)^{-1}$$

$$f'(x) = (-1)(1-2x)^{-2}(-2)$$

$$f''(x) = (-1)(-2)(1-2x)^{-3}(-2)^2$$

$$f'''(x) = (-1)(-2)(-3)(1-2x)^{-4}(-2)^3$$

$$\vdots$$

$$f^{(n)}(x) = (-1)(-2)\cdots(-n)(1-2x)^{-(n+1)}(-2)^n$$

$$\Rightarrow f^{(n)}(0) = n!\, 2^n$$

Thus

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{n!\, 2^n}{n!} x^n = \sum_{n=0}^{\infty} 2^n \cdot x^n$$

as before.

b) Use your result in a) to find the Taylor series for

$$g(x) = \frac{x}{1+2x^3}$$

again centered at $a = 0$.

$$g(x) = \frac{x}{1+2x^3} = x \cdot \frac{1}{1-2(-x^3)} = x \cdot f(-x^3) = x \cdot \sum_{n=0}^{\infty} 2^n (-x^3)^n = x \sum_{n=0}^{\infty} 2^n \cdot (-1)^n x^{3n} = \sum_{n=0}^{\infty} (-1)^n 2^n x^{3n+1}$$

(6) a) Use the method of separation of variables to find an equation which implicitly defines all solutions of the following differential equation

$$\frac{dy}{dx} = y^2 e^{3x}$$

This D.E. is separable. Thus

STEP 1 $y^2 = 0$

i.e. $y = 0$ is the only constant solution.

STEP 2/3 when $y \neq 0$,

$$\frac{1}{y^2}\frac{dy}{dx} = e^{3x}$$

$$\Rightarrow \int \frac{1}{y^2} dy = \int e^{3x} dx + C$$

$$\Rightarrow -\frac{1}{y} = \frac{1}{3}e^{3x} + C$$

b) Find an explicit expression for the solution of the above differential equation which satisfies $y(0) = 1$.

We can solve the above explicitly for $y$:

$$-\frac{1}{y} = \frac{1}{3}e^{3x} + C \Rightarrow -1 = \left(\frac{1}{3}e^{3x} + C\right) y$$

$$\Rightarrow y = \frac{-1}{\frac{1}{3}e^{3x} + C} = \frac{-3}{e^{3x} + 3C}$$

Using the initial condition, we find that

$$1 = y(0) = \frac{-3}{1 + 3C} \Rightarrow 1 + 3C = -3$$

$$3C = -4$$

$$\Rightarrow y(x) = \frac{-3}{e^{3x} - 4} = \frac{3}{4 - e^{3x}}.$$