1. On Induction

In mathematics, we are often faced with the challenge of proving infinitely many statements. Although such a task seems daunting, there is a particular form of logic which is very well suited for this kind of problem. In words, this is the logic of an inductive proof, or said differently, a proof by induction.

Lets begin with an example.

Example 1. *Prove that for any $n \in \mathbb{N}$, and any choice of complex numbers $z_1, z_2, \cdots, z_n \in \mathbb{C}$, we have that*

$$\overline{\sum_{k=1}^{n} z_k} = \sum_{k=1}^{n} \overline{z_k}\,. \tag{1}$$

In words, this example asks us to check that the complex conjugate of a sum is the sum of the corresponding complex conjugates; no matter how many terms there are in the particular sum.

Note that the claim in (1) is actually infinitely many different statements because it makes an assertion for each natural number $n \in \mathbb{N} = \{1, 2, 3, \cdots, \}$. One could try to verify this case by case, i.e. check the validity of the statement for $n = 1$, then $n = 2$, then $n = 3$, etc. Unfortunately, this would take forever, and one simply doesn't have the time. . .

This is where the logic of a proof by inductions comes in very handy.

Let us first describe the method of proof by induction, and then apply it to verify the claim in (1).

Proof by Induction: Suppose you want to prove infinitely many mathematical assertions. The method of proof by induction has three essential pieces. Let us describe each of these in detail.

(1) *Label the Assertions:* The first step in a proof by induction is to label the mathematical assertions that one wants to prove. In words, this step asks you to organize your thoughts and label the statements you want to prove. Abstractly, we can say for each $n \in \mathbb{N}$, let $A(n)$ describe the n-th mathematical assertion. Typically, in class or in a homework problem, this step has already been done for you. In the example above, the equation (1) is a labeling of the particular mathematical assertion of interest, i.e. that equation is $A(n)$.
(2) *Verify the Base Case:* The next step in a proof by induction is to check and see whether or not the first assertion (specifically, $A(1)$) is true. In words, this step is a reality check: If you want to prove infinitely many statements, the first one better be true . . .
(3) *Make an Inductive Argument:* The last step in a proof by induction requires that you make an inductive argument. The inductive argument can be stated as follows: Let $N \in \mathbb{N}$. Suppose you know that $A(N)$ is true. Use this information to verify that the statement $A(N + 1)$ is also true.

If you complete all steps described above you have proven that the assertions $A(n)$ are true for all natural numbers $n \in \mathbb{N}$, and you have done so with a proof by induction.

Let us now return to the first example above. For convenience, I repeat it.

Example 1. *Prove that for any $n \in \mathbb{N}$, and any choice of complex numbers $z_1, z_2, \cdots, z_n \in \mathbb{C}$, we have that*

$$\overline{\sum_{k=1}^{n} z_k} = \sum_{k=1}^{n} \overline{z_k}\,. \tag{2}$$

We now prove this by induction.

First, we need to label the statements. Let us declare that the assertions of interest, i.e. $A(n)$ is the statement of equation (10). *Typically, for homework, you do not have to tell me what the assertions are . . .*

Next, we need to verify the base case, i.e. $A(1)$. Let $z_1 \in \mathbb{C}$. we want to show that

$$\overline{\sum_{k=1}^{1} z_k} = \sum_{k=1}^{1} \overline{z_k}\,.$$

In some examples, the base case is trivial; like the above. There is nothing wrong with this, but it is still an important part of the proof by induction. To verify the above, we check that the right-hand-side (RHS) is equal to the left-hand-side (LHS).

The RHS is:

$$\sum_{k=1}^{1} \overline{z_k} = \overline{z_1}$$

The LHS is:

$$\overline{\sum_{k=1}^{1} z_k} = \overline{z_1}$$

Since they are indeed equal, we have verified that assertion $A(1)$ is true. This completes the step of verifying the base case.

Now we want to make the inductive argument. Let $N \in \mathbb{N}$. Suppose we know that $A(N)$ is true. Let us now check that $A(N+1)$ is true. Now, the statement $A(N)$ says that: Let $z_1, z_2, \cdots, z_N \in \mathbb{C}$, then

$$\overline{\sum_{k=1}^{N} z_k} = \sum_{k=1}^{N} \overline{z_k}\,. \tag{3}$$

Assuming the above, i.e. (3), is true, let us now try to verify that $A(N+1)$ is true. To this end, let $z_1, z_2, \cdots, z_N, z_{N+1} \in \mathbb{C}$. To ease notation, let us introduce

$$z = \sum_{k=1}^{N} z_k \quad \text{and} \quad w = z_{N+1}\,.$$

With this notation in mind, now consider the sum:

$$\overline{\sum_{k=1}^{N+1} z_k} = \overline{\sum_{k=1}^{N} z_k + z_{N+1}} = \overline{z+w} = \overline{z} + \overline{w} \tag{4}$$

where the last step above is a fact we proved in class; namely that $\overline{z+w} = \overline{z} + \overline{w}$ is true for any two complex numbers. By our inductive assumption, i.e. $A(N)$, we know that

$$\overline{z} = \overline{\sum_{k=1}^{N} z_k} = \sum_{k=1}^{N} \overline{z_k}$$

by (3). This prove that

$$\overline{\sum_{k=1}^{N+1} z_k} = \overline{z} + \overline{w} = \sum_{k=1}^{N} \overline{z_k} + \overline{z_{N+1}} = \sum_{k=1}^{N+1} \overline{z_k} \tag{5}$$

and thereby verifies that $A(N+1)$ is true.

This completes the proof by induction.

Let's do one more example.

Example 2. *Prove that for any $n \in \mathbb{N}$, and any choice of complex numbers $z_1, z_2, \cdots, z_n \in \mathbb{C}$, we have that*

$$\overline{\prod_{k=1}^{n} z_k} = \prod_{k=1}^{n} \overline{z_k}\,. \tag{6}$$

As before, the first step is to label the statements. Again, we will declare that the assertions of interest, i.e. $A(n)$ is the statement contained in equation (6).

Next, we need to verify the base case, i.e. $A(1)$. Let $z_1 \in \mathbb{C}$. we want to show that

$$\overline{\prod_{k=1}^{1} z_k} = \prod_{k=1}^{1} \overline{z_k}\,.$$

To verify the above, we check that the right-hand-side (RHS) is equal to the left-hand-side (LHS).

The RHS is:

$$\prod_{k=1}^{1} \overline{z_k} = \overline{z_1}$$

The LHS is:

$$\overline{\prod_{k=1}^{1} z_k} = \overline{z_1}$$

Since they are indeed equal, we have verified that assertion $A(1)$ is true. This completes the step of verifying the base case.

Now we want to make the inductive argument. Let $N \in \mathbb{N}$. Suppose we know that $A(N)$ is true. Let us now check that $A(N+1)$ is true. Now, the statement $A(N)$ says that: Let $z_1, z_2, \cdots, z_N \in \mathbb{C}$, then

$$\overline{\prod_{k=1}^{N} z_k} = \prod_{k=1}^{N} \overline{z_k}\,. \tag{7}$$

Assuming the above, i.e. (7), is true, let us now try to verify that $A(N+1)$ is true. To this end, let $z_1, z_2, \cdots, z_N, z_{N+1} \in \mathbb{C}$. To ease notation, let us now introduce

$$z = \prod_{k=1}^{N} z_k \quad \text{and} \quad w = z_{N+1}\,.$$

With this notation in mind, now consider the sum:

$$\overline{\prod_{k=1}^{N+1} z_k} = \overline{z_{N+1} \prod_{k=1}^{N} z_k} = \overline{wz} = \overline{w}\,\overline{z} \tag{8}$$

where the last step above is a fact we proved in class; namely that $\overline{wz} = \overline{w}\,\overline{z}$ is true for any two complex numbers. By our inductive assumption, i.e. $A(N)$, we know that

$$\overline{z} = \overline{\prod_{k=1}^{N} z_k} = \prod_{k=1}^{N} \overline{z_k}$$

by (7). This prove that

$$\overline{\prod_{k=1}^{N+1} z_k} = \overline{w}\,\overline{z} = \overline{z_{N+1}} \prod_{k=1}^{N} \overline{z_k} = \prod_{k=1}^{N+1} \overline{z_k} \tag{9}$$

and thereby verifies that $A(N+1)$ is true.

This completes the proof by induction.

Here is an example for you to try.

Example 3. *Prove that for any $n \in \mathbb{N}$, and any choice of complex numbers $z_1, z_2, \cdots, z_n \in \mathbb{C}$, we have that*

$$\left| \prod_{k=1}^{n} z_k \right| = \prod_{k=1}^{n} |z_k| \,. \tag{10}$$