MATH 396T
TEST 1: MAKE-UP
SIMS

SPRING 2020

Name	
I.D. Number	

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
Total	40	

Directions: This work is an optional assignment for those who took the first test on Monday, February 24th. It is due on Friday, March 6th at the beginning of class. No late work will be accepted. If you turn this in, I will grade it and your new grade on test 1 will be a weighted average (out of 100%) of the two scores you have received. (I will weight the make-up twice as much as the in-class test.) If you do not turn this in, your grade on test 1 will stay the same.

(1) Use induction to prove the following. Let $n \in \mathbb{N}$. Show that

$$\prod_{j=1}^{n} \left(1 - \frac{1}{(j+1)^2}\right) = \frac{n+2}{2(n+1)}$$

Recall that the notation on the right-hand-side above describes a product. In fact, when $n = 3$ that product is:

$$\prod_{j=1}^{3} \left(1 - \frac{1}{(j+1)^2}\right) = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right) = \left(\frac{3}{4}\right)\left(\frac{8}{9}\right)\left(\frac{15}{16}\right)$$

(2) Consider two complex numbers:

$$w = \frac{e^{4+5i}}{3-2i} \quad \text{and} \quad z = \frac{4+5i}{e^{3-2i}}$$

a) Find $\mathrm{Re}[w]$ and $\mathrm{Im}[w]$.

b) Find $\mathrm{Re}[z]$ and $\mathrm{Im}[z]$.

c) Find $|w|$ and $|z|$.

(3) Let f be an arithmetic function with period $q \in \mathbb{N}$. Show that for any integer $a \in \mathbb{Z}$,

$$q \sum_{n=1}^{q} f(n)\overline{f(n-a)} = \sum_{k=1}^{q} e(ak/q)|\hat{f}(k)|^2 \,.$$

State carefully any results you use in verifying this equality.

(4) Consider the arithmetic function f with period 3 satisfying

$$f(0) = -1, \quad f(1) = 2, \quad \text{and} \quad f(2) = -3\,.$$

i) Evaluate explicitly all three values of $e(-\frac{n}{3})$ with $n = 0, 1, 2$.

ii) Consider the arithmetic function g defined by setting $g(n) = \overline{f(n)}$ for all $n \in \mathbb{Z}$. (Here f is the function with period 3 defined above.) Find the Discrete Fourier Transform of g, i.e. find $\hat{g}(k)$ for all $k \in \mathbb{Z}$.