Key to Make Up

1. Use induction to prove that:

For any $n \in \mathbb{N}$,

$$(*) \quad \prod_{j=1}^{n} \left(1 - \frac{1}{(j+1)^2}\right) = \frac{n+2}{2(n+1)}$$

<u>Check Base case:</u>

$n=1$

LHS $\quad \prod_{j=1}^{1} \left(1 - \frac{1}{(j+1)^2}\right) = 1 - \frac{1}{2^2} = \frac{3}{4}$ ✓

RHS $\quad \frac{1+2}{2(1+1)} = \frac{3}{4}$ ✓

<u>Inductive step:</u>

Suppose we know $(*)$ for $n=N$.
Consider

$$\prod_{j=1}^{N+1} \left(1 - \frac{1}{(j+1)^2}\right) = \left(1 - \frac{1}{(N+2)^2}\right) \cdot \prod_{j=1}^{N} \left(1 - \frac{1}{(j+1)^2}\right)$$

$$= \left(\frac{(N+2)^2 - 1}{(N+2)^{\cancel{2}}}\right)\left(\frac{\cancel{N+2}}{2(N+1)}\right)$$

Note that

$$(N+2)^2 - 1 = N^2 + 4N + 4 - 1$$
$$= N^2 + 4N + 3$$
$$= (N+3)(N+1)$$

$$\Rightarrow \prod_{j=1}^{N+1}\left(1 - \frac{1}{(j+1)^2}\right) = \frac{\left((N+2)^2 - 1\right)}{(N+2)} \cdot \frac{1}{2(N+1)}$$

$$= \frac{(N+3)\cancel{(N+1)}}{2(N+2)\cancel{(N+1)}} = \frac{(N+1)+2}{2((N+1)+1)} \checkmark$$

This completes the proof.

2) Consider

$$w = \frac{e^{4+5i}}{3-2i} = \frac{e^4 \cdot e^{5i}}{(3-2i)} \frac{(3+2i)}{(3+2i)}$$

$$= \frac{e^4\left(\cos(5) + i\sin(5)\right)(3+2i)}{9+4}$$

$$= \frac{e^4}{13}\left(\left(3\cos(5) - 2\sin(5)\right) + i\left(2\cos(5) + 3\sin(5)\right)\right)$$

We conclude that

a) $$\mathrm{Re}[w] = \frac{e^4}{13}\left(3\cos(5) - 2\sin(5)\right)$$

and

$$\mathrm{Im}[w] = \frac{e^4}{13}\left(2\cos(5) + 3\sin(5)\right).$$

Now consider

$$z = \frac{4+5i}{e^{3-2i}} = (4+5i)\,e^{-3+2i}$$

$$= e^{-3}(4+5i)\left(\cos(2) + i\sin(2)\right)$$

$$= e^{-3}\left[\left(4\cos(2) - 5\sin(2)\right) + i\left(4\sin(2) + 5\cos(2)\right)\right]$$

b) $$\mathrm{Re}[z] = e^{-3}\left(4\cos(2) - 5\sin(2)\right)$$

and

$$\mathrm{Im}[z] = e^{-3}\left(4\sin(2) + 5\cos(2)\right)$$

c) $$|w| = \left|\frac{e^{4+5i}}{3-2i}\right| = \frac{|e^4|\cdot|e^{5i}|}{|3-2i|} = \frac{e^4}{\sqrt{13}}$$

$$|z| = \left|\frac{4+5i}{e^{3-2i}}\right| = \frac{|4+5i|}{|e^3|\cdot|e^{-2i}|} = e^{-3}\sqrt{41}$$

3) Let f be an arithmetic function of period $q \in \mathbb{N}$. Show that for any $a \in \mathbb{Z}$,

$$q \sum_{n=1}^{q} f(n) \overline{f(n-a)} = \sum_{k=1}^{q} e\left(\frac{ak}{q}\right) |\hat{f}(k)|^2.$$

Here we use Theorem 2.7

It says if f and g are arithmetic functions of period q, then

$$\sum_{n=1}^{q} f(n) \overline{g(n)} = \frac{1}{q} \sum_{k=1}^{q} \hat{f}(k) \overline{\hat{g}(k)}.$$

In this application, we will take f to be the function of period q and g to be the function

$$g(n) = f(n-a).$$

Note: If f has period q and $a \in \mathbb{Z}$, then g has period q.

Also note that by Theorem 2.4 a) (take $a = -a$ in statement)

$$\hat{g}(k) = \hat{f}(k)\, e\left(\frac{k(-a)}{q}\right)$$

We conclude that

$$\sum_{n=1}^{q} f(n)\overline{f(n-a)} = \sum_{n=1}^{q} f(n)\overline{g(n)}$$

$$\xrightarrow{\text{theorem 2.7}} = \frac{1}{q}\sum_{k=1}^{q} \hat{f}(k)\overline{\hat{g}(k)}$$

$$\xrightarrow{\text{theorem 2.4a)}} = \frac{1}{q}\sum_{k=1}^{q} \hat{f}(k)\overline{\left(\hat{f}(k)\, e\left(\frac{-ak}{q}\right)\right)}$$

$$= \frac{1}{q}\sum_{k=1}^{q} \hat{f}(k)\overline{\hat{f}(k)}\, e\left(\frac{ak}{q}\right)$$

$$= \frac{1}{q}\sum_{k=1}^{q} e\left(\frac{ak}{q}\right)|\hat{f}(k)|^2$$

Multiplying by q, we get the desired result.

---

4 i) $e\left(-\frac{0}{3}\right) = e(0) = 1$

$$e\left(-\frac{1}{3}\right) = e^{2\pi i\left(-\frac{1}{3}\right)} = \cos\left(\frac{-2\pi}{3}\right) + i\sin\left(\frac{-2\pi}{3}\right)$$

$$\xrightarrow{\text{use that cosine and sine are even and odd}} = \cos\left(\frac{2\pi}{3}\right) - i\sin\left(\frac{2\pi}{3}\right)$$

$$= -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) \Rightarrow \cos\left(2\frac{\pi}{3}\right) = \cos^2\left(\frac{\pi}{3}\right) - \sin^2\left(\frac{\pi}{3}\right)$

$\sin(2\theta) = 2\sin\theta\cos\theta \qquad \sin\left(2\frac{\pi}{3}\right) = 2\sin(\pi/3)\cos(\pi/3)$

Since $\sin(\pi/3) = \frac{\sqrt{3}}{2}$ and $\cos(\pi/3) = \frac{1}{2}$

we find that:

$$\cos(\tfrac{2\pi}{3}) = \cos(\pi/3)^2 - \sin(\pi/3)^2 = \left(\tfrac{1}{2}\right)^2 - \left(\tfrac{\sqrt{3}}{2}\right)^2$$

$$= \frac{1}{4} - \frac{3}{4} = \frac{-2}{4} = \frac{-1}{2}$$

$$\sin(\tfrac{2\pi}{3}) = 2\sin(\pi/3)\cos(\pi/3) = 2 \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{2} = \frac{\sqrt{3}}{2}$$

$$e(-\tfrac{2}{3}) = e^{2\pi i(\frac{-2}{3})} = \cos(\tfrac{4\pi}{3}) - i\sin(\tfrac{4\pi}{3})$$
$$= -1/2 + i\frac{\sqrt{3}}{2}$$

$$\cos(\tfrac{4\pi}{3}) = \cos^2(\tfrac{2\pi}{3}) - \sin^2(\tfrac{2\pi}{3}) = \left(\tfrac{-1}{2}\right)^2 - \left(\tfrac{\sqrt{3}}{2}\right)^2$$

$$= \frac{1}{4} - \frac{3}{4} = -\frac{1}{2}$$

$$\sin(\tfrac{4\pi}{3}) = 2\sin(\tfrac{2\pi}{3})\cos(\tfrac{2\pi}{3}) = 2\left(\tfrac{\sqrt{3}}{2}\right)\left(-\tfrac{1}{2}\right) = -\frac{\sqrt{3}}{2}$$

---

ii) Since f is real valued, $g(n) = \overline{f(n)} = f(n)$.

Thus

$$\hat{g}(0) = \hat{f}(0) = \sum_{n=0}^{2} f(n) = f(0) + f(1) + f(2)$$
$$= -1 + 2 - 3$$
$$= -2$$

$$\hat{g}(1) = \hat{f}(1) = \sum_{n=0}^{2} f(n)e(\tfrac{-n}{3}) = f(0)e(\tfrac{-0}{3}) + f(1)e(\tfrac{-1}{3}) + f(2)e(\tfrac{-2}{3})$$
$$= -1 + 2e(\tfrac{-1}{3}) - 3e(\tfrac{-2}{3})$$

4.

$$\hat{g}(1) = -1 + 2\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) - 3\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

$$= -1 - 1 + \frac{3}{2} - i\sqrt{3} - \frac{3i\sqrt{3}}{2}$$

$$= -\frac{1}{2} - \frac{5i\sqrt{3}}{2}$$

$$\hat{g}(2) = \hat{f}(2) = \sum_{n=0}^{2} f(n)\, e\left(-\frac{2n}{3}\right)$$

$$= f(0)\left(e\left(\frac{-0}{3}\right)\right)^2 + f(1)\left(e\left(-\frac{1}{3}\right)\right)^2 + f(2)\left(e\left(\frac{-2}{3}\right)\right)^2$$

$$= -1 \cdot (1) + 2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) - 3\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^2$$

$$= -1 - 1 + i\sqrt{3} - 3\left(\frac{1}{4} - \frac{2i\sqrt{3}}{4} - \frac{3}{4}\right)$$

$$= -2 + i\sqrt{3} - 3\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)$$

$$= -\frac{1}{2} + \frac{5i\sqrt{3}}{2}$$