MATH 425A
SOME BASIC AXIOMS AND THEIR CONSEQUENCES

FALL 2019

The foundation of analysis is the set of real numbers.

We will denote by $\mathbb{R}$ this set of real numbers, i.e.

$$\mathbb{R} = \{x \,|\, x \text{ is real}\}$$

For real numbers there are two basic operations: addition and multiplication. A notation for these operations is introduced as follows:

To each pair of real numbers x and y, there is a unique real number, which we denote by $x+y$, and refer to as the *sum* of x and y. This operation defines addition.

To each pair of real numbers x and y, there is a unique real number, which we denote by xy, and refer to as the *product* of x and y. This operation defines multiplication.

The set of real numbers $\mathbb{R}$ equipped with these two operations satisfy the field axioms. They are:

Axiom 1 (Field Axioms)**.** *Let $\mathbb{R}$ be the set of real numbers.*

- **Commutativity of Addition:** *For any $x, y \in \mathbb{R}$,*

$$x + y = y + x \,.$$

- **Associativity of Addition:** *For any $x, y, z \in \mathbb{R}$,*

$$(x + y) + z = x + (y + z) \,.$$

- **The Additive Identity:** *There is a real number, denoted by $0 \in \mathbb{R}$, for which*

$$0 + x = x + 0 = x \quad \textit{for all } x \in \mathbb{R} \,.$$

- **The Additive Inverse:** *For each real number $x \in \mathbb{R}$ there is a real number $y \in \mathbb{R}$ for which*

$$x + y = 0 \,.$$

- **Commutativity of Multiplication:** *For any $x, y \in \mathbb{R}$,*

$$xy = yx \,.$$

- **Associativity of Multiplication:** *For any $x, y, z \in \mathbb{R}$,*

$$(xy)z = x(yz) \,.$$

- **The Multiplicative Identity:** *There is a real number, denoted by* $1 \in \mathbb{R}$*, for which*

$$1x = x1 = x \quad \textit{for all } x \in \mathbb{R}\,.$$

- **The Multiplicative Inverse:** *For each real number* $x \neq 0$*, there is a real number* $y \in \mathbb{R}$ *for which*

$$xy = 1\,.$$

- **The Distributive Property:** *For any* $x, y, z \in \mathbb{R}$*,*

$$x(y + z) = xy + xz\,.$$

- **Nontriviality:**

$$1 \neq 0\,.$$

Consequences of the Field Axioms:

• The additive identity, labeled 0 above, is unique.

• For any $x \in \mathbb{R}$,

$$0x = x0 = 0$$

• For any $x, y \in \mathbb{R}$, if $xy = 0$, then either $x = 0$ or $y = 0$ (both is allowed).

• For any $a \in \mathbb{R}$, there is a unique solution of the equation

$$a + x = 0$$

The solution, which we denote by $x = -a$, is the additive inverse of a.

• For any $x, y \in \mathbb{R}$, the difference of x and y, which we denote by $x - y$ is defined by

$$x - y = x + (-y)$$

This is how *subtraction* is defined.

• For any $a \in \mathbb{R}$, one has that

$$-(-a) = a$$

• The multiplicative identity, labeled 1 above, is unique.

• For any $a \in \mathbb{R} \setminus \{0\}$, there is a unique solution of the equation

$$ax = 1$$

The solution, which we denote by $x = a^{-1} = \frac{1}{a}$, is the multiplicative inverse of a (also called the reciprocal of a).

• For any $x, y \in \mathbb{R}$ with $y \neq 0$, the quotient of x and y, which we denote by x/y (or $\frac{x}{y}$) is defined by

$$\frac{x}{y} = xy^{-1}$$

This is how *division* is defined.

• For any $a \in \mathbb{R} \setminus \{0\}$, one has that

$$(a^{-1})^{-1} = a$$

• For any $a \in \mathbb{R} \setminus \{0\}$, one has that

$$(-a)^{-1} = -a^{-1}$$

• By induction, one can prove: Let $a \in \mathbb{R}$ and $n \in \mathbb{N}$. For any $x_1, x_2, \cdots, x_n \in \mathbb{R}$,

$$a \sum_{k=1}^{n} x_k = \sum_{k=1}^{n} a x_k$$

Axiom 2 (Positivity Axioms)**.** *There is a subset of $\mathbb{R}$, denoted by $\mathcal{P}$, called the set of positive numbers for which:*

- *If x and y are positive, then $x + y$ and xy are both positive.*
- *For each $x \in \mathbb{R}$, exactly one of the following 3 alternatives is true:*
  (1) $x \in \mathcal{P}$,
  (2) $-x \in \mathcal{P}$,
  (3) $x = 0$.

Consequences of the Positivity Axioms:

• Let $x, y \in \mathbb{R}$. We write $x > y$ if and only if $x - y$ is positive. If $x > y$ we say that x is strictly greater than y. We write $x \geq y$ if and only if $x > y$ or $x = y$. If $x \geq y$ we say that x is greater than or equal to y.

• Let $x, y \in \mathbb{R}$. We write $x < y$ if and only if $y > x$. If $x < y$ we say x is strictly less than y. We write $x \leq y$ if and only if $x < y$ or $x = y$. If $x \leq y$ we say that x is less that or equal to y.

• For any $a \in \mathbb{R} \setminus \{0\}$, one has that

$$a^2 > 0$$

Since $1 \neq 0$, an application of this to $a = 1$ shows that $1 > 0$.

• For any $a > 0$, one has that

$$a^{-1} > 0$$

• Let $x, y \in \mathbb{R}$ with $x \leq y$. For any $c \in \mathbb{R}$,

$$x + c \leq y + c\,.$$

If the assumed inequality is strict, the resulting inequality is strict as well.

• Let $x_1, x_2, y_1, y_2 \in \mathbb{R}$ with $x_1 \leq y_1$ and $x_2 \leq y_2$. Then,

$$x_1 + x_2 \leq y_1 + y_2 \,.$$

If either of the assumed inequalities is strict, then the resulting inequality is strict as well.

An immediate consequence of the above is the following.

• Let $x_1, x_2, y_1, y_2 \in \mathbb{R}$ with $x_1 \leq y_1$ and $x_2 \geq y_2$. Then,

$$x_1 - x_2 \leq y_1 - y_2 \,.$$

If either of the assumed inequalities is strict, then the resulting inequality is strict as well.

• If $x > y$, then

$$xc > yc \quad \text{if } c > 0$$

and

$$xc < yc \quad \text{if } c < 0$$

• By induction, one can prove: For any $n \in \mathbb{N}$, let x_1, x_2, $\cdots$, x_n be non-negative numbers, i.e. $x_k \geq 0$ for all $k \in \{1, \cdots, n\}$.
One has that the sum of non-negative numbers is non-negative, i.e.

$$\sum_{k=1}^{n} x_k \geq 0$$

and moreover,

$$\sum_{k=1}^{n} x_k = 0 \quad \text{if and only if} \quad x_1 = x_2 = \cdots = x_n = 0.$$

One has that the product of non-negative numbers is non-negative, i.e.

$$x_1 x_2 \cdots x_n \geq 0$$

and moreover,

$$x_1 x_2 \cdots x_n = 0 \quad \text{if and only if} \quad \text{there is some } k \in \{1, \cdots, n\} \text{ for which } x_k = 0.$$

Chains of Inequalities:

It is sometimes useful to make statements involving multiple inequalities.
A valid chain of inequalities (with two links) is a statement of the form:
Let $x, y, z \in \mathbb{R}$.
We write

$$x \leq y \leq z \quad \text{if and only if} \quad x \leq y \quad \text{and} \quad y \leq z$$

In the case above, one checks that $x \leq z$.

We write

$$x \leq y < z \quad \text{if and only if} \quad x \leq y \quad \text{and} \quad y < z$$

In the case above, one checks that $x < z$.

We write

$$x < y \leq z \quad \text{if and only if} \quad x < y \quad \text{and} \quad y \leq z$$

In the case above, one checks that $x < z$.

More valid chains of inequalities are:
$x \geq y \geq z$, $x \geq y > z$, and $x > y \geq z$.

They are defined and have consequences similar to the above statements. These are the only valid chains of inequalities with two links. No other combination has a logical interpretation.

One can extend this notion to chains of inequalities with more than two links. The only valid chains are those for which:
- all linking inequalities are either $\geq$ or $>$.
- all linking inequalities are either $\leq$ or $<$.

No other combinations have a logical interpretation.