(1) Recall: we write $a \leq b \iff a = b$ or $a < b$

Moreover, $a = b \iff a - b = 0$

and $a < b \iff b - a > 0$.

Case 1: Suppose $x_1 = y_1$ and $x_2 = y_2$.

Calculate:

$$(y_1 + y_2) - (x_1 + x_2) = (y_1 + y_2) + (-x_1 - x_2)$$
$$= y_1 + (y_2 + (-x_1 - x_2))$$
$$= y_1 + (-x_1 + (y_2 - x_2))$$
$$= (y_1 - x_1) + (y_2 - x_2)$$
$$= 0 + 0 = 0$$

$\Rightarrow y_1 + y_2 = x_1 + x_2$ ✓

Case 2: $x_1 = y_1$ and $x_2 < y_2$.

Calculate:

as above

$$(y_1 + y_2) - (x_1 + x_2) = (y_1 - x_1) + (y_2 - x_2)$$
$$= 0 + (y_2 - x_2) > 0.$$

Thus $x_1 + x_2 < y_1 + y_2$.

Case 3: $x_1 < y_1$ and $x_2 = y_2$

calculate:

$$(y_1 + y_2) - (x_1 + x_2) = (y_1 - x_1) + (y_2 - x_2)$$
$$= y_1 - x_1 + 0 > 0.$$

Thus $x_1 + x_2 < y_1 + y_2$.

Case 4: $x_1 < y_1$ and $x_2 < y_2$:

$$(y_1 + y_2) - (x_1 + x_2) = (y_1 - x_1) + (y_2 - x$$

> 0 by pos. axiom

(2) Let $S(n)$ be the inequality

$$(1+a)^n \geq 1+na+\frac{n(n-1)}{2}a^2 \qquad \text{for } a \geq 0 \text{ and } n \in \mathbb{N}.$$

<u>Base case</u> $S(1)$

LHS $\quad (1+a)^1 = 1+a$

RHS $\quad 1+(1)a+\frac{(1)(1-1)}{2}a^2 = 1+a$ ✓

<u>Suppose $S(n)$ is true:</u>

<u>Then</u> $\quad (1+a)^n \geq 1+na+\frac{n(n-1)}{2}a^2$

Since $(1+a) \geq 0$ we can multiply on both sides above:

$$(1+a)^{n+1} = (1+a)^n(1+a) \geq \left(1+na+\frac{n(n-1)}{2}a^2\right)(1+a)$$

$$= 1+na+\frac{n(n-1)}{2}a^2+a+na^2+\frac{n(n-1)}{2}a^3$$

$$= 1+(n+1)a+\left[\frac{n(n-1)}{2}+n\right]a^2+\frac{n(n-1)}{2}a^3$$

$$= 1+(n+1)a+\left(\frac{n^2-n+2n}{2}\right)a^2+\frac{n(n-1)}{2}a^3$$

$\boxed{\text{since } \frac{n(n-1)}{2}a^3 \geq 0}$

$$\geq 1+(n+1)a+\frac{(n+1)n}{2}a^2$$

This is the claim.

(3)

a) Let $S(n)$ be the statement that $x_n > 0$ for $n \in \mathbb{N}$.

Base case: $S(1)$ $x_1 = 1 > 0$ ✓.

Inductive step: Suppose $S(n)$ is true, i.e. $x_n > 0$.

By positivity axioms: $4 + x_n > 0$, $5 + x_n > 0$, and $(5+x_n)^{-1} > 0$.

Then

$$x_{n+1} = \frac{4+x_n}{5+x_n} = (4+x_n)(5+x_n)^{-1} > 0 \text{ by positivity.}$$

This proves a).

b) Let $S(n)$ be the statement that $x_n - x_{n+1} \geq 0$ for $n \in \mathbb{N}$.

Base Case: $S(1)$ $x_1 = 1$ $x_2 = \frac{4+1}{5+1} = \frac{5}{6} \Rightarrow x_1 - x_2 = 1 - \frac{5}{6} = \frac{1}{6} > 0$. ✓

Inductive Step

$$x_{n+2} - x_{n+1} = \frac{4+x_{n+1}}{5+x_{n+1}} - \frac{4+x_n}{5+x_n} = \frac{(4+x_{n+1})(5+x_n) - (4+x_n)(5+x_{n+1})}{(5+x_{n+1})(5+x_n)}$$

This proves b).

$$= \frac{x_n - x_{n+1}}{(5+x_n)(5+x_{n+1})} \geq 0$$

c) Since $\{x_n\}$ is positive and monotonically decreasing:

$0 < x_n \leq x_1$ and hence $\{x_n\}$ is bounded.

By Monotone Convergence, $x = \lim_{n\to\infty} x_n$ exists.

Since $\{x_{n+1}\}$ is a subsequence of $\{x_n\}$ we know that

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{4+x_n}{5+x_n} = \frac{4+x}{5+x}.$$

$$\Rightarrow x(5+x) = 4+x \Rightarrow x^2 + 4x - 4 = 0 \Rightarrow x_\pm = \frac{-4 \pm 4\sqrt{2}}{2}$$

Since $x_n \geq 0$ for all $n \in \mathbb{N}$, $x \geq 0$ by Lemma.

Hence $x = x_+ = -2 + 2\sqrt{2} > 0$.

(4) a) Let $x_0 \in \mathbb{R}$. Since $\mathbb{Q} \subset \mathbb{R}$ is dense, there is a sequence $\{x_n\}$ in $\mathbb{Q}$ with $\{x_n\}$ converging to x_0. Since f is continuous, f is continuous at x_0 and thus

$$f(x_0) = \lim_{n \to \infty} f(x_n).$$

Since $x_n \in \mathbb{Q}$ for all $n \in \mathbb{N}$, $f(x_n) = 0$ for all $n \in \mathbb{N}$.

Thus $f(x_0) = 0$ as the limit of a constant sequence.

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b) Define $g : [-a, a] \to \mathbb{R}$ by setting.

$$g(x) = f(x) - x.$$

Since $h(x) = -x$ is a polynomial, h is continuous.
Since $g(x) = f(x) + h(x)$ and f and h are continuous, then g is continuous.

- $g(-a) = f(-a) + a > 0$
- $g(a) = f(a) - a < 0$

Thus by the Intermediate Value Theorem, there is $x \in (-a, a)$ for which $g(x) = 0$ i.e.

$$f(x) - x = 0$$

i.e. $f(x) = x$ for this particular $x \in (-a, a)$.