MATH 425A
EXAM 2

FALL 2019

Show all work. You may use any result proven in class, or homework problem, but you must state the results you are using precisely.

(1) a) Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = x^2 + 4 \quad \text{for all } x \in \mathbb{R}\,.$$

Is f uniformly continuous? Justify your answer.

b) Consider the function $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = \frac{1}{x^2 + 4} \quad \text{for all } x \in \mathbb{R}\,.$$

Is g uniformly continuous? Justify your answer.

(2) a) Let $g : \mathbb{R} \to \mathbb{R}$ be differentiable with $g'(x) > 0$ for all $x \in \mathbb{R}$ and moreover, suppose $g(\mathbb{R}) = \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Define $h : \mathbb{R} \to \mathbb{R}$ by setting

$$h(x) = f(g^{-1}(x)) \quad \text{for all } x \in \mathbb{R}\,.$$

Show that h is differentiable and find $h'(x)$ for all $x \in \mathbb{R}$.

b) Let $x_0 \in \mathbb{R}$ and suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable at x_0. Determine whether the following limit exists:

$$\lim_{x \to x_0} \frac{x f(x_0) - x_0 f(x)}{x - x_0}$$

If the limit exists, find its value. If not, explain. In either case, justify your answer.

(3) Consider the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x & \text{if } x \in [0,1] \text{ is rational.} \\ 0 & \text{if } x \in [0,1] \text{ is irrational.} \end{cases}$$

Prove that f is not integrable by showing that

$$\underline{\int_0^1} f = 0 \quad \text{and} \quad \overline{\int_0^1} f \geq 1/2.$$

(4) Let $f : [0,1] \to \mathbb{R}$ be continuous and monotonically increasing. Let $g : [1,2] \to \mathbb{R}$ be continuous and monotonically decreasing. Define $h : [0,2] \to \mathbb{R}$ by setting

$$h(x) = \begin{cases} f(x) & \text{if } x \in [0,1), \\ 0 & \text{if } x = 1, \\ g(x) & \text{if } x \in (1,2]. \end{cases}$$

Prove that h is integrable on $[0,2]$ by finding an Archimedean sequence for h on $[0,2]$. Express the integral of h in terms of the integrals of f and g. Justify your claims.