Key to Exam 2

1) a) No, $f(x)=x^2+4$ is not uniformly continuous on $\mathbb{R}$.

Consider two sequences $\{u_n\}$ and $\{v_n\}$ with

$u_n = n+\frac{1}{n}$ and $v_n = n$ for all $n \in \mathbb{N}$.

Clearly

$u_n - v_n = \frac{1}{n}$ and so $\lim_{n\to\infty} u_n - v_n = 0$.

Also

$f(u_n) = (n+\frac{1}{n})^2+4 = n^2+\frac{1}{n^2}+6$ and $f(v_n)=n^2+4$

Thus

$f(u_n)-f(v_n) = 2+\frac{1}{n^2}$ and so $\lim_{n\to\infty} f(u_n)-f(v_n) = 2 \neq 0$.

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1) b) yes $g(x)=\frac{1}{x^2+4}$ is uniformly continuous on $\mathbb{R}$.

Let $x,y\in\mathbb{R}$, Note that

$$g(x)-g(y) = \frac{1}{x^2+4} - \frac{1}{y^2+4} = \frac{(y^2+4)-(x^2+4)}{(x^2+4)(y^2+4)} = \frac{y^2-x^2}{(x^2+4)(y^2+4)}$$

Thus

$$|g(x)-g(y)| = \frac{|x-y|\cdot|x+y|}{(x^2+4)(y^2+4)} \le |x-y|\left(\frac{|x|}{(x^2+4)}+\frac{|y|}{(y^2+4)}\right)$$

Cauchy $\neq$ $$\le |x-y|\left(\frac{1}{2}\frac{(x^2+1)}{(x^2+4)}+\frac{1}{2}\frac{(y^2+1)}{(y^2+4)}\right)$$

$$\le |x-y|.$$

Thus for all $\varepsilon>0$ there is $\delta=\varepsilon>0$ s.t.

$|g(x)-g(y)|<\varepsilon$ if $|x-y|<\delta=\varepsilon$.

2a) Since $g'(x) > 0$ for all $x \in \mathbb{R}$, we have a result (Corollary 4.21) which shows that g is strictly increasing. Since g is strictly increasing with non-zero derivative, we have a result (Theorem 4.11) which shows that g^{-1} is differentiable. In fact, by Corollary 4.1[illegible]

$$(g^{-1})'(x) = \frac{1}{g'(g^{-1}(x))}.$$

Since f and g^{-1} are both differentiable, the chain rule (Theorem 4.14) applies and so $h = f \circ g^{-1}$ is differentiable. Moreover,

$$h'(x) = f'(g^{-1}(x))(g^{-1})'(x) = f'(g^{-1}(x)) \cdot \frac{1}{g'(g^{-1}(x))}$$

$$= \left(\frac{f'}{g'}\right)(g^{-1}(x)).$$

2b) Note that for any $x \neq x_0$, the quantity of interest is:

$$\frac{x f(x_0) - x_0 f(x)}{x - x_0} = \frac{(x - x_0 + x_0) f(x_0) - x_0 f(x)}{x - x_0}$$

$$= f(x_0) + \frac{x_0 (f(x_0) - f(x))}{x - x_0}$$

$$= f(x_0) - x_0 \frac{(f(x) - f(x_0))}{x - x_0}$$

Thus the limit exists and

$$\lim_{x \to x_0} \frac{x f(x_0) - x_0 f(x)}{x - x_0} = f(x_0) - x_0 f'(x_0)$$

since f is differentiable at x_0, i.e. $f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$.

3) We 1^{st} prove that $\underline{\int_0^1} f = 0$.

Recall: $\underline{\int_0^1} f = \sup\{L(f,P) \mid P \text{ is a partition of } [0,1]\}$.

We prove that $L(f,P) = 0$ for all partitions P. The above follows.

Let P be a partition of $[0,1]$.

$L(f,P) = \sum_{i=1}^{n} m_i (x_i - x_{i-1})$ where $m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}$.

Since $f(x) \geq 0$ for all $x \in [0,1]$, each $m_i \geq 0$.

Since irrational numbers are dense, there is $x_i^* \in [x_{i-1}, x_i] \cap (\mathbb{R} \setminus \mathbb{Q})$.

Thus $m_i \leq f(x_i^*) = 0 \Rightarrow m_i = 0$ for all $1 \leq i \leq n$ and hence,

$L(f,P) = 0$.

We next prove that $\overline{\int_0^1} f \geq 1/2$.

Recall: $\overline{\int_0^1} f = \inf\{U(f,P) \mid P \text{ is a partition of } [0,1]\}$.

Let $g : [0,1] \to \mathbb{R}$ be the function $g(x) = x$.

Since g is increasing, we know g is integrable on $[0,1]$.

As proven in class, the regular partitions $\{P_n\}$ are Archimedean for g and so

$$\overline{\int_0^1} g = \int_0^1 g = \lim_{n\to\infty} U(g, P_n) = \lim_{n\to\infty} \sum_{i=1}^{n} M_i(g)(x_i - x_{i-1})$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} x_i \cdot \frac{1}{n}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{i}{n} = \lim_{n\to\infty} \frac{1}{n^2} \frac{n(n+1)}{2} = \frac{1}{2}$$

Claim: For all partitions P of $[0,1]$,

$U(f,P) = U(g,P)$.

In this case, $\overline{\int_0^1} f = \overline{\int_0^1} g = 1/2$.

$$U(f,P) = \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1})$$

$$= \sum_{i=1}^{n} x_i (x_i - x_{i-1})$$

$$= U(g,P)$$

Here we use that:

$$M_i(f) = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\} = x_i$$

Note $\quad f(x) \le x_i$ for all $x \in [x_{i-1}, x_i]$.

Since rational numbers are dense, there is a sequence $\{x_n\} \subset [x_{i-1}, x_i] \cap \mathbb{Q}$ with $x_n \to x_i$

Thus $\quad x_n = f(x_n) \le M_i(f) \le x_i$ ✓

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4) · Since $f: [0,1] \to \mathbb{R}$ is increasing, the sequence of regular partitions $\{P_n^{[0,1]}\}$ is Archimedean for f.

· Since $g: [1,2] \to \mathbb{R}$ is decreasing, the sequence of regular partitions $\{P_n^{[1,2]}\}$ is Archimedean for g.

Claim: $\{P_n\}$ with $P_n = P_n^{[0,1]} \cup P_n^{[1,2]}$ is Archimedean for h.

Note: $$U(h, P_n) = U(f, P_n^{[0,1]}) + U(g, P_n^{[1,2]}) + \frac{(M_n(h) - M_n(f))}{n} + \frac{(M_1(h) - M_1(g))}{n}$$

also
$$L(h, P_n) = L(f, P_n^{[0,1]}) + L(g, P_n^{[1,2]}) + \frac{(m_n(h) - m_n(f))}{n} + \frac{(m_1(h) - m_1(g))}{n}$$

Since f, g, and h are bounded,

$$U(h, P_n) - L(h, P_n) = \left(U(f, P_n^{[0,1]}) - L(f, P_n^{[0,1]})\right) + \left(U(g, P_n^{[1,2]}) - L(g, P_n^{[1,2]})\right) + R_n$$

$$\Rightarrow \lim_{n \to \infty} \left(U(h, P_n) - L(h, P_n)\right) = 0 \quad \text{since} \quad |R_n| \le \frac{8M}{n}$$ ← bound on all functions

Moreover

$$\int_0^2 h = \lim_{n\to\infty} U(h, P_n)$$

$$= \lim_{n\to\infty}\left(U(f, P_n^{[0,1]}) + U(g, P_n^{[1,2]}) + \frac{M_n(h)-M_n(f)}{n} + \frac{M_1(h)-M_1(g)}{n} \right)$$

$$= \int_0^1 f + \int_1^2 g + 0.$$