# MATH 425A:
# EXAM 2
# MAKE-UP

FALL 2019

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|---|---|
| Name | |
| I.D. Number | |

| Question | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| Total | 40 | |

**Directions:** This work is an optional assignment for those who took the second test on Friday, November 15th. It is due on Wednesday, November 27th at the beginning of class. No late work will be accepted. If you turn this in, I will grade it and your new grade on Exam 2 will be the average (out of 100%) of the two scores you have received. If you do not turn this in, your grade on Exam 2 will stay the same.

Show all work. You may use any result proven in class, or homework problem, but you must state the results you are using precisely.

(1) Let $f : (a, b) \to \mathbb{R}$ be differentiable and $f' : (a, b) \to \mathbb{R}$ be continuous. Let $[c, d] \subset (a, b)$. Show that for every $\epsilon > 0$ there is a $\delta > 0$ for which:

$$|f(x) - f(y) - f'(y)(x - y)| < \epsilon$$

whenever $x, y \in [c, d]$ satisfy

$$|x - y| < \delta.$$

This is the notion of *uniformly differentiable.*

(2) a) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Determine whether or not

$$\lim_{x \to 0} \frac{f(x^2) - f(0)}{x}$$

exists. If it does, find its value. If it does not, explain.

b) Consider the function $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = \begin{cases} x - x^2 & \text{if } x \in \mathbb{Q}, \\ x + x^2 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Determine whether $g'(0)$ exists. If it exists, determine whether $g$ is increasing, decreasing, or constant in a neighborhood of 0. Explain your answer carefully.

(3) a) Let $\{P_n\}$ be a sequence of partitions of $[a, b]$. We say that $\{\tilde{P}_n\}$ is a *sequence of refinements* of $\{P_n\}$ if: for each $n \in \mathbb{N}$, $\tilde{P}_n$ is a refinement of $P_n$. Let $f : [a, b] \to \mathbb{R}$ be integrable. Let $\{P_n\}$ be an Archimedean sequence for $f$ on $[a, b]$. Prove that every sequence of refinements $\{\tilde{P}_n\}$ of $\{P_n\}$ is also Archimedean for $f$ on $[a, b]$.

b) Prove that

$$f(x) = \begin{cases} x - 1 & \text{if } x \in [2, 3], \\ 4 & \text{if } x \in (3, 4] \end{cases}$$

is integrable just using Upper and Lower Darboux sums. Find the value of the integral.

(4) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Let $a : \mathbb{R} \to \mathbb{R}$ be differentiable and $b : \mathbb{R} \to \mathbb{R}$ be differentiable. Define $F : \mathbb{R} \to \mathbb{R}$ by setting

$$F(x) = \int_{a(x)}^{b(x)} f(t)\,dt \quad \text{for all } x \in \mathbb{R}.$$

Determine whether $F$ is differentiable and if so, find its derivative.