Exam 2 MakeUp Key

1). Let $f : (a,b) \to \mathbb{R}$ be differentiable and $f' : (a,b) \to \mathbb{R}$ be continuous. Let $[c,d] \subset (a,b)$. Show that for each $\varepsilon > 0$, there is $\delta > 0$ such that

$$|f(x) - f(y) - f'(y)(x-y)| < \varepsilon$$

whenever $x, y \in [c,d]$ satisfy

$$|x-y| < \delta.$$

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proof:

Since $f : (a,b) \to \mathbb{R}$ is differentiable, $f : (a,b) \to \mathbb{R}$ is also continuous. In this case, for each $[c,d] \subset (a,b)$, $f : [c,d] \to \mathbb{R}$ is continuous and also uniformly continuous. Thus for every $\varepsilon > 0$, there is $\delta_1 > 0$ for which

$$|f(x) - f(y)| < \varepsilon/2$$

whenever $x, y \in [c,d]$ satisfy

$$|x-y| < \delta_1$$

Since $f':(a,b) \to \mathbb{R}$ is continuous,
$f':[c,d] \to \mathbb{R}$ is continuous as well.
Moreover $f':[c,d] \to \mathbb{R}$ is bounded.
Thus there is $M>0$ for which

$$|f'(x)| \leq M \quad \text{for all } x \in [c,d]$$

Now let $\varepsilon > 0$. Take $\delta_2 > 0$ so that $\delta_2 = \frac{\varepsilon}{2M}$
with $\delta = \min(\delta_1, \delta_2) > 0$ and $x, y \in [c,d]$
$|x-y| < \delta$ means that

$$|f(x) - f(y) - f'(y)(x-y)|$$

$$\leq |f(x) - f(y)| + |f'(y)| \cdot |x-y|$$

$$\leq \frac{\varepsilon}{2} + M \cdot \delta$$

$$\leq \frac{\varepsilon}{2} + M \cdot \delta_2$$

$$\leq \varepsilon$$

2) a) Let $f: \mathbb{R} \to \mathbb{R}$ be differentiable. Determine whether or not the limit

$$\lim_{x \to 0} \frac{f(x^2) - f(0)}{x}$$ exists.

b) Let $g: \mathbb{R} \to \mathbb{R}$ be given by

$$g(x) = \begin{cases} x - x^2 & \text{if } x \in \mathbb{Q} \\ x + x^2 & \text{if } x \in (\mathbb{R}/\mathbb{Q}). \end{cases}$$

Find $g'(0)$. Discuss behavior of g in a neighborhood of 0.

2a) Proof:

Set $g: \mathbb{R} \to \mathbb{R}$ to be

$g(x) = x^2$.

we know that g is differentiable and $g'(x) = 2x$.

In this case, $h: \mathbb{R} \to \mathbb{R}$ defined by setting

$h(x) = (f \circ g)(x)$

is also differentiable by the chain rule.

In this case,

$$h'(0) = \lim_{x \to 0} \frac{h(x) - h(0)}{x - 0} \quad \text{exists}$$

$$= \lim_{x \to 0} \frac{(f \circ g)(x) - (f \circ g)(0)}{x}$$

$$= \lim_{x \to 0} \frac{f(x^2) - f(0)}{x}$$

By the chain rule, we also know that

$$h'(x) = (f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

$$= 2x \cdot f'(x^2)$$

$$\Rightarrow \quad h'(0) = 2 \cdot 0 \cdot f'(0) = 0.$$

b)

$$g'(0) = \lim_{x \to 0} \frac{g(x) - g(0)}{x - 0}$$

Since $0 \in \mathbb{Q}$, $g(0) = 0 - 0^2 = 0$.

~~If $x \in \mathbb{Q}$~~,

$$\frac{g(x)}{x} = \begin{cases} \frac{x - x^2}{x} & \text{if } x \in \mathbb{Q} \setminus \{0\} \\ \frac{x + x^2}{x} & \text{if } x \in \mathbb{R} / \mathbb{Q} \end{cases}$$

$$= \begin{cases} 1 - x & \text{if } x \in \mathbb{Q} \setminus \{0\} \\ 1 + x & \text{if } x \in \mathbb{R} / \mathbb{Q} \end{cases}$$

In this case,

$$\lim_{x\to 0} \frac{g(x)}{x} = 1 \quad \text{i.e.} \quad g'(0) = 1 > 0.$$

From calculus, one might expect that there is a neighborhood of 0 on which f is increasing, but this is not the case.

Note that:

If $x \in \mathbb{R} \setminus \mathbb{Q}$ and $y \in \mathbb{Q}$, then.

$$g(y) - g(x) = (y - y^2) - (x + x^2)$$

$$= (y - x) - (x^2 + y^2)$$

Let $n \in \mathbb{N}$. Take $y_n = \frac{1}{2n}$ Take $x_n \in$ ~~$\left(\frac{1}{2^n}, \frac{1}{2^n}\right) \cap \mathbb{Q}$~~ $\left(\frac{1}{2n} - \frac{1}{2^{2n}}, \frac{1}{2n}\right) \cap \mathbb{Q}$

Then $x_n < y_n$. and

$$g(y_n) - g(x_n) = (y_n - x_n) - (x_n^2 + y_n^2)$$

$$\leq \cancel{\left(\frac{1}{2n} - \frac{1}{2^{2n}}\right) - \left(\frac{1}{2^{4n}} + \frac{1}{2^{2n}}\right)}$$

$$\leq \frac{1}{2^{2n}} - \left(x_n^2 + \frac{1}{2^{2n}}\right)$$

$$= -x_n^2 < 0$$

g is decreasing along this sequence ...

3a) Let $\{P_n\}$ be a sequence of partitions of $[a,b]$. We say that $\{\tilde{P}_n\}$ is a sequence of refinements of $\{P_n\}$ if: For each $n \in \mathbb{N}$, $\tilde{P}_n$ is a refinement of P_n. Let $f: [a,b] \to \mathbb{R}$ be integrable. Let $\{P_n\}$ be an Archimedean sequence for f on $[a,b]$. Prove that every sequence of refinements $\{\tilde{P}_n\}$ of $\{P_n\}$ is also Archimedean for f on $[a,b]$.

b) Consider $f: [2,4] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x-1 & \text{if } 2 \le x \le 3, \\ 4 & \text{if } 3 < x \le 4. \end{cases}$$

Show that f is integrable and find the value of $\int_2^4 f(x)\,dx$.

3a) Proof. Since $\{P_n\}$ is Archimedean for f on $[a,b]$, we know that

$$(*) \quad \lim_{n\to\infty}\Big(U(f,P_n) - L(f,P_n)\Big) = 0.$$

If $\{\tilde{P}_n\}$ is a sequence of refinements of $\{P_n\}$, then

For each $n \in \mathbb{N}$,

$$L(f, P_n) \leq L(f, \tilde{P}_n) \quad \text{and} \quad U(f, \tilde{P}_n) \leq U(f, P_n)$$

by the Refinement Lemma.

In this case, for each $n \in \mathbb{N}$

$$0 \leq U(f, \tilde{P}_n) - L(f, \tilde{P}_n) \leq U(f, P_n) - L(f, P_n)$$

Thus by comparison,

$$\lim_{n \to \infty} \left(U(f, \tilde{P}_n) - L(f, \tilde{P}_n) \right) = 0$$

using (*). This proves that $\{\tilde{P}_n\}$ is Archimedean for f on $[a,b]$.

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b) For each $n \in \mathbb{N}$, let

P_n be a regular partition of $[2,3]$

and Q_n be a regular partition of $[3,4]$.

Then

$P_n = \{x_j\}_{j=0}^{n}$ with $x_j = 2 + \frac{j}{n}$ for $0 \leq j \leq n$

and

$Q_n = \{y_j\}_{j=0}^{n}$ with $y_j = 3 + \frac{j}{n}$ for $0 \leq j \leq n$.

Take $\tilde{P}_n = P_n \cup Q_n$ which is a partition of $[2,4]$.

Note that

$$U(f,\hat{P}_n) = U(f,P_n) + U(f,Q_n)$$

and

$$L(f,\hat{P}_n) = L(f,P_n) + L(f,Q_n)$$

we show the 1st, the 2nd is similar.

Write $\hat{P}_n = \{z_j\}_{j=0}^{2n}$ where $z_j = \begin{cases} x_j & 0 \le j \le n \\ y_{j-n} & n+1 \le j \le 2n \end{cases}$

Pf $\boxed{z_n = x_n = y_0 = 3}$

$$U(f,\hat{P}_n) = \sum_{j=1}^{2n} M_j(f)(z_j - z_{j-1})$$

$$= \sum_{j=1}^{n} M_j(f)(x_j - x_{j-1})$$

$$+ \sum_{j=n+1}^{2n} M_{n+j}(f)(y_j - y_{j-1})$$

$\boxed{\text{Note } y_0 = x_n = z_n}$

$$= U(f,P_n) + U(f,Q_n)$$

The rest follows similarly.

Thm

$$0 \le U(f,\hat{P}_n) - L(f,\hat{P}_n)$$

$$= \big(U(f,P_n) - L(f,P_n)\big) + \big(U(f,Q_n) - L(f,Q_n)\big)$$

Now

$$U(f,P_n) - L(f,P_n) = \sum_{j=1}^{n} \left(M_j(f) - m_j(f)\right)(x_j - x_{j-1})$$

$$= \frac{1}{n} \sum_{j=1}^{n} \left((x_j - 1) - (x_{j-1} - 1)\right) \cdot 1$$

$$= \frac{1}{n} \sum_{j=1}^{n} (x_j - x_{j-1})$$

$$= \frac{1}{n} \cdot \sum_{j=1}^{n} \frac{1}{n} = \frac{1}{n}$$

and

$$U(f,Q_n) - L(f,Q_n) = \sum_{j=1}^{n} \left(M_j(f) - m_j(f)\right)(y_j - y_{j-1})$$

$$= \left(M_1(f) - m_1(f)\right) \cdot \frac{1}{n} + \cancel{\sum_{j=2}^{n} 0 \cdot \frac{1}{n}}^{\,0}$$

$$= (4-2) \cdot \frac{1}{n} = \frac{2}{n}.$$

Thus $\lim_{n\to\infty}\left(U(f,\tilde{P}_n) - L(f,\tilde{P}_n)\right) = \lim_{n\to\infty}\left(U(f,P_n) - L(f,P_n)\right)$

$$+ \lim_{n\to\infty}\left(U(f,Q_n) - L(f,Q_n)\right)$$

$$= \lim_{n\to\infty} \frac{1}{n} + \lim_{n\to\infty} \frac{2}{n} = 0 + 0 = 0.$$

Thus $\{\tilde{P}_n\}$ is Archimedean for f on $[2,4]$.

Moreover

$$\int_2^4 f = \lim_{n\to\infty} U(f,\tilde{P}_n) = \lim_{n\to\infty}\left(U(f,P_n) + U(f,Q_n)\right)$$

$$= \lim_{n\to\infty}\left(\sum_{j=1}^{n} (x_j - 1)(x_j - x_{j-1}) + \sum_{j=1}^{n} 4(x_j - x_{j-1})\right)$$

use that $\sum_{j}^{n} j = \frac{n(n+1)}{2}$

$$= \lim_{n\to\infty}\left(\sum_{j=1}^{n} \left(2 + \frac{j}{n} - 1\right)\left(\frac{1}{n}\right) + 4\sum_{j=1}^{n} \frac{1}{n}\right) = 5\tfrac{1}{2}.$$

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4) Let $f: \mathbb{R} \to \mathbb{R}$ be continuous.
Let $a, b: \mathbb{R} \to \mathbb{R}$ be differentiable.
Define $F: \mathbb{R} \to \mathbb{R}$ by setting.

$$F(x) = \int_{a(x)}^{b(x)} f(t)\,dt.$$

Show that F is differentiable and evaluate $F'(x)$.

~~proof:~~ Since $f: \mathbb{R} \to \mathbb{R}$ is continuous, ~~for each~~ define $G: \mathbb{R} \to \mathbb{R}$ by setting

$$G(x) = \int_{0}^{x} f(t)\,dt.$$

This is well defined since $f: [0, x] \to \mathbb{R}$ is continuous for each $x \geq 0$ and $f: [x, 0] \to \mathbb{R}$ is continuous for each $x < 0$.

Note that

$$F(x) = \int_{a(x)}^{b(x)} f(t)\,dt = \int_{0}^{b(x)} f(t)\,dt + \int_{a(x)}^{0} f(t)\,dt$$

$$= G(b(x)) - \int_{0}^{a(x)} f(t)\,dt$$

$$= G(b(x)) - G(a(x))$$

By the Fundamental theorem of calculus and the chain rule, both $G \circ a$ and $G \circ b$ are differentiable. Moreover,

$$F'(x) = (G \circ b)'(x) - (G \circ a)'(x)$$

$$= G'(b(x)) \cdot b'(x) - G'(a(x)) \cdot a'(x)$$

$$= f(b(x)) b'(x) - f(a(x)) \cdot a'(x).$$