1. Some Basics from Linear Algebra

First and foremost, I will assume that you are familiar with many basic facts about real and complex numbers. In particular, both $\mathbb{R}$ and $\mathbb{C}$ are fields; they satisfy the field axioms. For $z = x + iy \in \mathbb{C}$, the modulus, i.e. $|z| = \sqrt{x^2 + y^2} \geq 0$, represents the distance from z to the origin in the complex plane. (As such, it coincides with the absolute value for real z.) For $z = x + iy \in \mathbb{C}$, complex conjugation, i.e. $\overline{z} = x - iy$, represents reflection about the x-axis in the complex plane. It will also be important that both $\mathbb{R}$ and $\mathbb{C}$ are complete, as metric spaces, when equipped with the metric $d(z, w) = |z - w|$; more on this below.

1.1. Vector Spaces.

One of the most important preliminary notions for this course is that of a vector space. Although vector spaces can be defined over any field, we will restrict our attention to fields $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. In fact, quite often $\mathbb{F} = \mathbb{C}$ will be the field of choice.

The following definition is fundamental.

Definition 1.1. *Let $\mathbb{F}$ be a field. A* **vector space** V **over** $\mathbb{F}$ *is a non-empty set V (the elements of V are called* **vectors***) over a field $\mathbb{F}$ (the elements of $\mathbb{F}$ are called* **scalars***) equipped with two operations:*
i) To each pair $u, v \in V$, there exists a unique element $u + v \in V$. This operation is called **vector addition***.*
ii) To each $u \in V$ and $\alpha \in \mathbb{F}$, there exists a unique element $\alpha u \in V$. This operation is called **scalar multiplication***.*
These operations satisfy the following relations:
For all $\alpha, \beta \in \mathbb{F}$ and all $u, v, w \in V$,

1. $u + (v + w) = (u + v) + w$ *and* $u + v = v + u$
2. *There is a vector $0 \in V$ (called the* **additive identity***) such that $u + 0 = u$ for all $u \in V$*
3. *For each vector $u \in V$, there is a vector $-u \in V$ (called the* **additive inverse** *of u) such that $u + (-u) = 0$*
4. $\alpha(u + v) = \alpha u + \alpha v$
5. $(\alpha + \beta)u = \alpha u + \beta u$
6. $(\alpha\beta)u = \alpha(\beta u)$
7. $1u = u$ *for all* $u \in V$

The phrase "Let V be a complex (or real) vector space." means that V is a vector space over $\mathbb{F} = \mathbb{C}$ (or $\mathbb{F} = \mathbb{R}$). It is clear that every complex vector space is a real vector space.

Example 1 (Vectors)**.** *Let $\mathbb{F}$ be a field and $n \geq 1$ be an integer. Take*

$$V = \{(v_1, v_2, \cdots, v_n)^t : v_j \in \mathbb{F} \textit{ for all } 1 \leq j \leq n\}$$

The set V is often called the collection of n-tuples with entries in $\mathbb{F}$, and some write $V = \mathbb{F}^n$. Here the super script t indicated that I think of these n-tuples as columns; not rows. With the usual notions of addition and scalar multiplication, i.e. for $v, w \in V$ and $\lambda \in \mathbb{F}$, set

$$v + w = (v_1 + w_1, v_2 + w_2, \cdots, v_n + w_n)^t \quad \textit{and} \quad \lambda v = (\lambda v_1, \lambda v_2, \cdots, \lambda v_n)^t$$

V is a vector space over $\mathbb{F}$.

Example 2 (Matrices)**.** *Let $\mathbb{F}$ be a field and take integers $m, n \geq 1$. Take*

$$V = \{A = \{a_{ij}\} : a_{ij} \in \mathbb{F} \textit{ for all } 1 \leq i \leq m \textit{ and } 1 \leq j \leq n\}$$

The set V is often called the collection of $m \times n$ matrices with entries in $\mathbb{F}$, and some write $V = \mathbb{F}^{m \times n}$. Here we often visualize A as a matrix with m rows and n columns, i.e.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

With the usual notions of addition and scalar multiplication, i.e. for $A, B \in V$ and $\lambda \in \mathbb{F}$, set

$$A + B = \{c_{ij}\} \quad \text{with } c_{ij} = a_{ij} + b_{ij} \quad \text{for all } 1 \le i \le m \text{ and } 1 \le j \le n \quad \text{and} \quad \lambda A = \{\lambda a_{ij}\}$$

V is a vector space over $\mathbb{F}$.

One can argue that Example 2 is a special case of Example 1, however, it is often useful to think of these as two distinct examples . . .

Example 3 (Functions)**.** *Let $I \subset \mathbb{R}$ be an interval. Consider the set*

$$V = \{f : f : I \to \mathbb{F}\}$$

The set V is often called the collection of $\mathbb{F}$-valued functions on I, and some write $V = \mathcal{F}(I, \mathbb{F})$. With the usual notions of addition and scalar multiplication, i.e. for $f, g \in V$ and $\lambda \in \mathbb{F}$, set

$$(f + g)(x) = f(x) + g(x) \quad \text{and} \quad (\lambda f)(x) = \lambda f(x) \quad \text{for all } x \in I,$$

V is a vector space over $\mathbb{F}$.

Definition 1.2. *Let V be a vector space over $\mathbb{F}$. A non-empty set $U \subset V$ is said to be a subspace of V if U is a vector space over $\mathbb{F}$ when it is equipped with the same addition and scalar multiplication rules that make V a vector space over $\mathbb{F}$.*

To check that a (non-empty) subset $U \subset V$ is a subspace, one need only check closure under addition and scalar multiplication, i.e. $u, v \in U$ imply $u + v \in U$ and $u \in U$ imply $\lambda u \in U$ for all $\lambda \in \mathbb{F}$.

Let V be a vector space over $\mathbb{F}$ and $n \ge 1$ be an integer. Let $v_1, v_2, \cdots, v_n \in V$ and $\lambda_1, \lambda_2, \cdots, \lambda_n \in \mathbb{F}$. The vector $v \in V$ given by

$$v = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n = \sum_{i=1}^{n} \lambda_i v_i$$

is called a linear combination of the vectors $v_1, v_2, \cdots, v_n$.

Definition 1.3. *Let V be a vector space over $\mathbb{F}$. Let $n \ge 1$ and $v_1, v_2, \cdots, v_n \in V$. The collection of all linear combinations of the vectors $v_1, v_2, \cdots, v_n$, regarded as a subset of V, is called the span of these vectors. Our notation for this it*

$$\operatorname{span}(v_1, v_2, \cdots, v_n) = \{v = \sum_{i=1}^{n} \lambda_i v_i : \lambda_i \in \mathbb{F} \text{ for all } 1 \le i \le n\}$$

One readily checks that for any $n \ge 1$ and any collection of vectors $v_1, v_2, \cdots, v_n \in V$, $\operatorname{span}(v_1, v_2, \cdots, v_n) \subset V$ is a subspace of V. More generally, if $U \subset V$ is non-empty, then the linear hull $L(U)$ is the set of all (finite) linear combinations of elements of U. One checks that this is the smallest subspace of V containing U.

Definition 1.4. *Let V be a vector space over $\mathbb{F}$. If there is some $n \ge 1$ and vectors $v_1, v_2, \cdots, v_n \in V$ for which*

$$\operatorname{span}(v_1, v_2, \cdots, v_n) = V\,,$$

then V is said to be finite-dimensional. Any collection of vectors for which the above is true is called a spanning set for V. If V is not finite dimensional, then V is said to be infinite-dimensional.

Let us return to our examples.

Consider Example 1. The collection of n-tuples $\{e_j\}$ with $1 \leq j \leq n$ defined by $e_j = (0, 0, \cdots, 0, 1, 0, \cdots, 0)$ with the multiplicative identity $1 \in \mathbb{F}$ in the j-th component and the additive identitiy $0 \in \mathbb{F}$ in all other components is a spanning set for $V = \mathbb{F}^n$. In this case, V is finite-dimensional.

Consider Example 2. The collection of matrices $\{E_{ij}\}$ defined by fixing $1 \leq i \leq m$ and $1 \leq j \leq n$ and declaring that E_{ij} has a 1 in the i, j entry and 0 in all other entries has $mn < \infty$ elements. One checks that this is a spanning set for $V = \mathbb{F}^{m \times n}$, and hence V is finite-dimensional.

Consider Example 3. In general, this vector space is not finite-dimensional. In fact, take $\mathbb{F} = \mathbb{R}$ and $I = (0, 1)$. For any $n \geq 1$, one can construct n disjoint compact sub-intervals of $(0, 1/2)$. For each of these sub-intervals, choose a non-zero, function supported in that sub-interval. The span of these functions will clearly not include any function compactly supported in $(1/2, 1)$.

Definition 1.5. *Let V be a vector space over $\mathbb{F}$. A collection of vectors $v_1, v_2, \cdots, v_n \in V$ is said to be linearly independent if the only solution of the equation*

$$\sum_{i=1}^{n} \lambda_i v_i = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n = 0$$

with $\lambda_1, \lambda_2, \cdots, \lambda_n \in \mathbb{F}$ is $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$.

Definition 1.6. *Let V be a finite-dimensional vector space over $\mathbb{F}$. A collection of vectors $v_1, v_2, \cdots, v_n \in V$ is said to be a basis of V if the collection is a linearly independent, spanning set. In other words, the collection $v_1, v_2, \cdots, v_n$ is linearly independent and $\operatorname{span}(v_1, v_2, \cdots, v_n) = V$.*

One can prove that every finite dimensional vector space has a basis. One can also prove that for a fixed finite-dimensional vector space V, any two bases have the same number of elements.

Definition 1.7. *Let $V \neq \{0\}$ be a finite-dimensional vector space over $\mathbb{F}$. Denote by $\dim(V)$ the number of elements in any basis of V. This positive integer is called the dimension of V. By convention, we take $\dim(\{0\}) = 0$.*

Consider Example 1. The collection of n-tuples $\{e_j\}$, defined previously, is a basis of $V = \mathbb{F}^n$. As such, $\dim(\mathbb{F}^n) = n$.

Consider Example 2. The collection of matrices $\{E_{ij}\}$, defined previously, is a basis of $V = \mathbb{F}^{m \times n}$. As such, $\dim(\mathbb{F}^{m \times n}) = mn$.

1.2. **On distance, length, and angles.** We will frequently encounter sets equipped with certain structures. Three important examples follow.

1.2.1. *On distance.* Some sets are equipped with a function which describes distance. Such a function is called a metric. More precisely,

Definition 1.8. *A metric on a (non-empty) set X is a function $\rho : X \times X \to [0, \infty)$, with $(x, y) \mapsto \rho(x, y)$, which satisfies the following:*

1. *$\rho(x, y) = 0$ if and only if $x = y$;*
2. *$\rho(x, y) = \rho(y, x)$ for all $x, y \in X$;*
3. *$\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for all $x, y, z \in X$.*

A set X equipped with a metric is called a metric space; this is often written as (X, ρ).

If (X, ρ) is a metric space, then $\rho(x, y)$ is interpreted as the distance between x and y in X.

Note that, in general, a metric space need not have an additive structure. To be clear, given a metric space (X, ρ) and two points $x, y \in X$, it is not a priori clear that there is a well-defined sum of x and y, and in particular, a metric space need not be a vector space. What will be important for us is that, in the context of metric spaces, one can define *completeness*.

Definition 1.9. *Let (X, ρ) be a metric space. A sequence $\{x_n\}_{n=1}^{\infty}$ in X is said to converge to $x \in X$ if $\lim_{n\to\infty} \rho(x_n, x) = 0$. This may be written as $x_n \to x$. A sequence $\{x_n\}_{n=1}^{\infty}$ in X is said to be Cauchy if $\rho(x_n, x_m) \to 0$ as $m, n \to \infty$. The metric space (X, ρ) is said to be complete if every Cauchy sequence converges to an element of X.*

1.2.2. *On length.* Some sets are equipped with a function which describes length. Such a function is called a norm. For this notion, an additive structure is required. More precisely,

Definition 1.10. *Let V be a vector space over $\mathbb{F}$. A map: $\|\cdot\| : V \to \mathbb{R}$, $x \mapsto \|x\|$, is said to be a norm on V if it satisfies the following properties:*

(1) **Sub-additivity:** *For each $x, y \in V$,*
$$\|x + y\| \leq \|x\| + \|y\|.$$
(2) **Positive Homogeneity:** *For each $x \in V$ and $\lambda \in \mathbb{F}$,*
$$\|\lambda x\| = |\lambda|\|x\|.$$
(3) **Positive Definiteness:** *For each $x \in V$, $\|x\| \geq 0$ and moreover, $\|x\| = 0$ if and only if $x = 0$.*

V is said to be a normed space if it is a vector space equipped with a norm; this is often written $(V, \|\cdot\|)$.

If $(V, \|\cdot\|)$ is a normed space, then $\|x\|$ is interpreted as the length of x in X.

One can easily show that every normed space $(V, \|\cdot\|)$ is a metric space when V is equipped with the metric $\rho(x, y) = \|x - y\|$. This leads to an important definition.

Definition 1.11. *Let $(V, \|\cdot\|)$ be a normed space. If the corresponding metric space (V, ρ), with $\rho(x, y) = \|x - y\|$, is complete, then V is said to be a Banach space.*

In some cases, one encounters a function on a vector space which only satisfies sub-additivity and positive homogeneity. Such functions are called semi-norms. One can check that a semi-norm is a norm if and only if the semi-norm takes non-zero vectors to non-zero real numbers.

1.2.3. *On angles.* Some sets are equipped with a function which describes the angle between two elements. Such functions are called inner products or scalar products.

Definition 1.12. *Let V be a vector space over $\mathbb{F}$. A map: $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$, $(x, y) \mapsto \langle x, y \rangle$, is said to be an inner-product on V if it satisfies the following properties:*

(1) **Second Component Linear:** *For each $x, y, z \in V$ and any $\lambda \in \mathbb{F}$, one has that*
$$\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle \quad \textit{and} \quad \langle x, \lambda y \rangle = \lambda \langle x, y \rangle$$
(2) **Conjugate Symmetry:** *For each $x, y \in V$, $\langle x, y \rangle = \overline{\langle y, x \rangle}$.*
(3) **Positive Definiteness:** *For each $x \in V$, $\langle x, x \rangle \geq 0$ and moreover, $\langle x, x \rangle = 0$ if and only if $x = 0$.*

V is said to be an inner-product space (or pre-Hilbert Space) if it is a vector space equipped with an inner-product; this is often written $(V, \langle \cdot, \cdot \rangle)$.

In this class, we will mainly deal with complex inner-product spaces.

It is not hard to show that every inner-product space is a normed space. In fact, let $(V, \langle \cdot, \cdot \rangle)$ be an inner-product space. The map $\|\cdot\| : V \to [0, \infty)$ with
$$\|x\| = \sqrt{\langle x, x \rangle} \quad \text{for each } x \in V\,,$$
is easily checked to be a norm.

In fact, the first step in this proof is the following.

Theorem 1.13 (Cauchy-Schwarz Inequality)**.** *Let $(V, \langle \cdot, \cdot \rangle)$ be an inner-product space. For each $x, y \in V$, one has that*

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

where $\|x\| = \sqrt{\langle x, x \rangle}$ as discussed above.

Given the bound above, it is straight-forward to verify that every inner-product space is a normed space. As such, by a previous discussion, it is also a metric space. In this case, the notion of completeness is relevant. For clarity, the metric here is:

$$\rho(x, y) = \|x - y\| = \sqrt{\langle x - y, x - y \rangle} \quad \text{for all } x, y \in V$$

Definition 1.14. *Let $(V, \langle \cdot, \cdot \rangle)$ be an inner-product space. If (V, ρ), regarded as a metric space with metric as described above, is complete, then V is said to be a Hilbert space.*

Remark: Not all normed spaces are inner-product spaces. In fact, one can show that the norm on a normed space $(V, \| \cdot \|)$ arises from an inner-product if and only if the norm satisfies

$$\|x + y\|^2 + \|x - y\|^2 = 2 \left(\|x\|^2 + \|y\|^2 \right) \quad \text{for all } x, y \in V\,.$$

This well-known relation is called the Parallelogram Law.

In words, the above result shows that not all Banach spaces are Hilbert spaces. Examples of this include $L^1(\mathbb{R})$ and $C((0,1))$ equipped with $\| \cdot \|_\infty$. One can check that every finite dimensional inner-product space is a Hilbert space.

For a real inner-product space V, given two vectors $x, y \in V \setminus \{0\}$, the quantity

$$\theta_{x,y} = \cos^{-1} \left(\frac{\langle x, y \rangle}{\|x\| \|y\|} \right) \in [0, \pi] \tag{1}$$

is said to describe the angle between x and y in V.

1.2.4. *Some Examples.* Consider Example 1. One readily checks that

$$\langle x, y \rangle = \sum_{i=1}^{n} \overline{x_i} \cdot y_i$$

defines an inner-product on $V = \mathbb{C}^n$. In this case, $V = \mathbb{C}^n$ is a normed space when equipped with

$$\|x\| = \sqrt{\sum_{i=1}^{n} |x_i|^2}$$

Consider Example 2. One readily checks that

$$\langle A, B \rangle_{\text{HS}} = \text{Tr}[A^* B]$$

defines an inner-product on $V = \mathbb{C}^{n \times n}$. This is called the Hilbert-Schmidt inner product. In this case, $V = \mathbb{C}^{n \times n}$ is a normed space when equipped with

$$\|A\|_{\text{HS}} = \sqrt{\text{Tr}[A^* A]}$$

which is called the Hilbert-Schmidt norm.

Consider Example 2 again. One readily checks that $V = \mathbb{C}^{n \times n}$ is a normed space when equipped with

$$\|A\| = \sup_{\substack{\psi \in \mathbb{C}^n: \\ \psi \neq 0}} \frac{\|A\psi\|}{\|\psi\|}$$

This norm is called the operator norm.

Consider Example 3. For any $f \in V$, one can define

$$\|f\|_\infty = \sup_{x \in I} |f(x)|\,.$$

One can check that

$$U = \{f \in V : \|f\|_\infty < \infty\}$$

is a subspace of V. In fact, one checks that $\|\cdot\|_\infty$ is a norm on U.

Consider Example 3 again. For any measurable $f \in V$, one can define

$$\|f\|_2 = \sqrt{\int_I |f(x)|^2 dx}\,.$$

One can check that

$$\tilde{U} = \{f \in V : f \text{ is measurable and } \|f\|_2 < \infty\}$$

is a subspace of V. In fact, one checks that $\|\cdot\|_2$ is a semi-norm on $\tilde{U}$. What's more, one can further check that $\|\cdot\|_2$ is the semi-norm induced by the semi-inner-product

$$\langle f, g\rangle = \int_I \overline{f(x)} \cdot g(x)\, dx\,.$$

An easy way to make this semi-norm into a norm is to first restrict to continuous functions $f \in V$. In this case, the the semi-inner-product also becomes an inner-product.

Remark: We have to work harder to get a Hilbert space out of $\|\cdot\|_2$. This requires equivalence classes of functions which agree almost everywhere. The resulting Hilbert space is denoted by $L^2(I)$.