1. On Orthogonality

The notion of orthogonal, or perpendicular, vectors in an inner-product space is quite useful. Here we introduce some simple consequences.

Definition 1.1. *Let $\mathcal{H}$ be a pre-Hilbert space. (Thus $\mathcal{H}$ is a vector space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ equipped with an inner-produce $\langle \cdot, \cdot \rangle$. The use of the pre-fix* **pre** *indicates that this inner-product space need not be complete.) Two vectors $f, g \in \mathcal{H}$ are said to be orthogonal if $\langle f, g \rangle = 0$. We may write this as $f \perp g$.*

Note that if $\mathcal{H}$ is a pre-Hilbert space and $f, g \in \mathcal{H}$ are orthogonal, i.e. $f \perp g$, then

$$\|f+g\|^2 = \langle f+g, f+g \rangle = \|f\|^2 + \|g\|^2$$

and this formula is often called the *Pythagorean Theorem* for obvious reasons.

Let us now fix a pre-Hilbert space $\mathcal{H}$. A vector $f \in \mathcal{H}$ is said to be *orthogonal to a subset* $U \subset \mathcal{H}$ if $\langle f, g \rangle = 0$ for all $g \in U$. This may be written as $f \perp U$. Two subsets $U, V \subset \mathcal{H}$ are said to be orthogonal, written $U \perp V$, if $\langle f, g \rangle = 0$ for all $f \in U$ and $g \in V$. If $U \subset \mathcal{H}$, then the set

$$U^{\perp} = \{f \in \mathcal{H} : f \perp U\}$$

is called the *orthogonal complement of U.*

The following proposition, see page 29 of the text, summarizes several useful properties associated with this notion of orthogonality.

Proposition 1.2. *Let $\mathcal{H}$ be a pre-Hilbert space.*

1. *One can check that $\{0\}^{\perp} = \mathcal{H}$ and $\mathcal{H}^{\perp} = \{0\}$. In words, this shows that 0 is the only vector orthogonal to every element of $\mathcal{H}$.*
2. *For every $U \subset \mathcal{H}$, $U^{\perp}$ is a closed subspace of $\mathcal{H}$.*
3. *If $U \subset V \subset \mathcal{H}$, then $V^{\perp} \subset U^{\perp}$.*
4. *For every $U \subset \mathcal{H}$,*

$$U^{\perp} = L(U)^{\perp} = \left(\overline{L(U)}\right)^{\perp}.$$

Note: In the above, for any $U \subset \mathcal{H}$, $L(U) \subset \mathcal{H}$ is the set of all finite linear combinations of elements of U. As such, it is the smallest subspace of $\mathcal{H}$ containing U. Thus by (2) above, we also know that $U^{\perp} = L(U^{\perp}) = \overline{L(U^{\perp})}$.

1.1. **On internal and external direct sums.**

A direct sum is a special form of the sum of two *sub-spaces.*

1.1.1. *On internal direct sums.*

Let $\mathcal{H}$ be a vector space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. For any two subspaces, $U_1, U_2 \subset \mathcal{H}$, the sum of these subspaces is given by

$$U_1 + U_2 = \{f \in \mathcal{H} : f = g_1 + g_2 \text{ where } g_1 \in U_1 \text{ and } g_2 \in U_2\}$$

One readily checks that $U_1 + U_2 \subset \mathcal{H}$ is a subspace.

In the special case that $U_1, U_2 \subset \mathcal{H}$ are subspaces with trivial intersection, i.e. $U_1 \cap U_2 = \{0\}$, then the sum of these subspaces is written as $U_1 \dot{+} U_2$ and such a sum is called a *direct sum.*

Direct sums are particularly useful because one readily checks that: If $U_1 \dot{+} U_2$ is the direct sum of subspaces in $\mathcal{H}$, then each $f \in U_1 \dot{+} U_2$ has a unique representation as $f = g_1 + g_2$ with $g_1 \in U_1$ and $g_2 \in U_2$.

If $\mathcal{H}$ is a pre-Hilbert space and $U_1, U_2 \subset \mathcal{H}$ are orthogonal subspaces, i.e. $U_1 \perp U_2$, then clearly $U_1 \cap U_2 = \{0\}$. In this case, the direct sum is written as $U_1 \oplus U_2$ and is called an *orthogonal sum.*

An important fact, which is a consequence of Theorem 3.3 c) on page 32, is the following. Let $\mathcal{H}$ be a Hilbert space and $U \subset \mathcal{H}$ be a closed subspace. Then $\mathcal{H} = U \oplus U^{\perp}$.

The above described *internal* direct sums, i.e. sums of subspaces of a fixed vector space.

1.1.2. *On external direct sums.* In some cases, we want to add two vector spaces together. Special cases of this form external direct sums.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be vector spaces over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. The Cartesian product of these spaces, i.e.

$$\mathcal{H}_1 \times \mathcal{H}_2 = \{(f, g) : f \in \mathcal{H}_1 \text{ and } g \in \mathcal{H}_2\}$$

is clearly a vector space over $\mathbb{F}$ with the usual notions of *vector* addition an scalar multiplication.

If $\mathcal{H}_1$ and $\mathcal{H}_2$ are both pre-Hilbert spaces over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, then $\mathcal{H}_1 \times \mathcal{H}_2$ is a pre-Hilbert space as well, when equipped with

$$\langle (f_1, g_1), (f_2, g_2) \rangle = \langle f_1, f_2 \rangle_1 + \langle g_1, g_2 \rangle_2$$

One checks that $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2$ is a Hilbert space if and only if $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces.

The Hilbert space $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2$ is often written as $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, and called an external direct sum. This is because we can identify $\mathcal{H}_1$ with the subspace

$$U_1 = \{(f, 0) : f \in \mathcal{H}_1\} \subset \mathcal{H}$$

and similarly identify $\mathcal{H}_2$ with the subspace

$$U_2 = \{(0, g) : g \in \mathcal{H}_2\} \subset \mathcal{H}$$

Since $U_1 \perp U_2$, this external direct sum may be identified with an internal direct sum in $\mathcal{H}$.

For more on this, see Exercise 1.11 on page 14, Exercise 2.2 a) on page 21, and the comments starting with *If A is an arbitrary set* ... on page 33.