Lecture 12

On C^*-algebras

In previous classes, we often discussed the collection of "observables" of a quantum system as the "algebra of observables" for a given quantum system. This phrase has meaning, and today we discuss this in detail.

Let us begin by observing important properties in one of the most important examples.

On $B(\mathcal{H})$:

Let $\mathcal{H}$ be a complex Hilbert space. By $B(\mathcal{H})$ we denote the collection of all bounded linear operators on $\mathcal{H}$. Recall that when $\mathcal{H}$ is finite dimensional, then $B(\mathcal{H})$ is just a collection of matrices.

- It is clear that $B(\mathcal{H})$ is a vector space.
- Moreover, $B(\mathcal{H})$ is a normed vector space when it is equipped with

$$\|A\| = \sup_{\psi \in \mathcal{H}/\{0\}} \frac{\|A\psi\|}{\|\psi\|} \quad \text{for all } A \in B(\mathcal{H}).$$

The quantity above is called the operator norm and $\|A\|$ is said to be the norm of A.

- In terms of this norm, $B(\mathcal{H})$ is also a metric space. In fact,

$$d(A,B) = \|A - B\| \quad \text{for all } A, B \in B(\mathcal{H})$$

defines a metric on $B(\mathcal{H})$.

- One readily checks that $B(\mathcal{H})$ is complete (as a metric space) with respect to this metric induced by the operator norm, and hence, $B(\mathcal{H})$ is a <u>Banach space</u>.

- There is also a well defined product on $B(\mathcal{H})$, it is just the composition of linear maps, and this product satisfies the following norm estimate:

$$\|AB\| \leq \|A\| \cdot \|B\| \quad \text{for all } A, B \in B(\mathcal{H}).$$

The above estimate is sufficient to imply that this product, i.e. the map $(A,B) \mapsto AB$, is norm continuous. These facts together imply that $B(\mathcal{H})$ is a <u>Banach algebra</u>.

It will also be important that there is a well-defined map $*: B(H) \to B(H)$ which satisfies certain properties. On $B(H)$, the $*$ map is the map $A \mapsto A^*$ the adjoint of $A \in B(H)$. Recall that the adjoint map satisfies the following properties:

- $(\alpha A + \beta B)^* = \bar{\alpha} A^* + \bar{\beta} B^*$ for all $A, B \in B(H)$ and $\alpha, \beta \in \mathbb{C}$.

(In words, the $*$-map is anti-linear.)

- $(A^*)^* = A$ for all $A \in B(H)$

(In words, the $*$-map is an involution.)

One also has that

- $(AB)^* = B^* A^*$ for all $A, B \in B(H)$
- $\|A^*\| = \|A\|$ for all $A \in B(H)$
- $\|A^*A\| = \|A\|^2$ for all $A \in B(H)$.

We will now declare that these properties we have seen in $B(H)$ are useful and important as a structure. In the language of mathematics, we will say that anything that satisfies all these properties is a C^*-algebra. This structure will play an important role later when we investigate thermodynamic limits.

Some general structure

<u>Definition 1</u> A complex vectorspace $\mathcal{Q}$ is said to be an <u>algebra</u> if it is equipped with a product map, i.e.

$$(\cdot,\cdot) : \mathcal{A} \times \mathcal{A} \to \mathcal{A} , \qquad (A,B) \mapsto AB ,$$ for which

i) $A(BC) = (AB)C$ for all $A, B, C \in \mathcal{A}$

ii) $A(B+C) = AB + AC$ for all $A, B, C \in \mathcal{A}$

iii) $\alpha\beta(AB) = (\alpha A)(\beta B)$ for all $A, B \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$.

In words, a vector space with a product is an <u>algebra</u>. Often the above is referred to as an associative algebra, but we will just say algebra.

Let $\mathcal{A}$ be an algebra. An element $\mathbb{1} \in \mathcal{A}$ is said to be an <u>identity element</u> if

$$\mathbb{1} A = A = A \mathbb{1} \quad \text{for all } A \in \mathcal{A}.$$

Any algebra $\mathcal{A}$ with an identity element is called a <u>unital</u> algebra.

A subspace $\mathcal{B} \subset \mathcal{A}$ that is also an algebra with respect to the operations in $\mathcal{A}$ is called a <u>subalgebra</u> of $\mathcal{A}$.

Definition 2: An algebra $\mathcal{A}$ is called a *-algebra if there is a map $* : \mathcal{A} \to \mathcal{A}$, $A \mapsto A^*$, for which

i) $(A^*)^* = A$ for all $A \in \mathcal{A}$ (i.e. $*$ is an involution)

ii) $(AB)^* = B^* A^*$ for all $A, B \in \mathcal{A}$ (i.e. $*$ is an anti-morphism)

iii) $(\alpha A + \beta B)^* = \bar{\alpha} A^* + \bar{\beta} B^*$ for all $A, B \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$

(i.e. $*$ is anti-linear)

Let $\mathcal{A}$ be a *-algebra. A subset $\mathcal{B} \subset \mathcal{A}$ is said to be self-adjoint if $A \in \mathcal{B} \Rightarrow A^* \in \mathcal{B}$.

Note: Self-adjoint subsets are subsets that are closed under the $*$ map.

An element $A \in \mathcal{A}$ is said to be self-adjoint if $A^* = A$.

If we denote by $\mathcal{A}_{sa} \subset \mathcal{A}$ the collection of all self-adjoint elements of $\mathcal{A}$, it is clear that $\mathcal{A}_{sa}$ is a self-adjoint subset of $\mathcal{A}$.

<u>Definition 3</u> An algebra $\mathfrak{a}$ is called a <u>normed algebra</u> if there is a map $\|\cdot\|: \mathfrak{a} \to \mathbb{R}$ for which

i) $\|A\| \geq 0$ for all $A \in \mathfrak{a}$ and $\|A\| = 0 \iff A = 0$.

ii) $\|\alpha A\| = |\alpha| \cdot \|A\|$ for all $A \in \mathfrak{a}$ and $\alpha \in \mathbb{C}$.

iii) $\|A+B\| \leq \|A\| + \|B\|$ for all $A, B \in \mathfrak{a}$.

iv) $\|AB\| \leq \|A\| \cdot \|B\|$ for all $A, B \in \mathfrak{a}$.

<u>Note</u>: Properties i), ii), and iii) above show that a normed algebra is an algebra with a norm on it. Property iv) shows that the norm "respects products" in the sense of this inequality.

Let $\mathfrak{a}$ be a normed algebra. For each $A \in \mathfrak{a}$, we call $\|A\|$ the norm of A. Since the norm on $\mathfrak{a}$ is a norm, it defines a metric as follows:

$$d(A, B) = \|A - B\| \quad \text{for all } A, B \in \mathfrak{a}.$$

This is said to be the metric induced by this norm.
This metric can be used to define open balls (and hence open sets). This collection of open sets is a topology.
It is called the <u>uniform</u> or <u>norm</u> topology on $\mathfrak{a}$.

Let $\mathfrak{A}$ be a normed algebra. If $\mathfrak{A}$ is complete (as a metric space) with respect to the metric induced by its norm, then $\mathfrak{A}$ is called a <u>Banach algebra</u>.

- A $*$-algebra $\mathfrak{A}$ is said to be a <u>normed</u> <u>$*$-algebra</u> if $\mathfrak{A}$ is a normed algebra and the norm satisfies

$$\|A^*\| = \|A\| \qquad \text{for all } A \in \mathfrak{A}.$$

(The above equality shows that the map $A \mapsto A^*$ is continuous in norm.)

- A normed $*$-algebra is said to be a <u>Banach $*$-algebra</u> if it is complete with respect to the metric induced by its norm.

We now present the main definition of the day.

<u>Definition</u> We say that a Banach $*$-algebra $\mathfrak{A}$ is a <u>C^*-algebra</u> if

$$\|A^*A\| = \|A\|^2 \quad \text{for all } A \in \mathfrak{A}.$$

The final equality above is often called the <u>C^*-property</u>.

Example 1 Let H be a complex Hilbert space. Let $\mathcal{A} = B(H)$ the collection of bounded linear operators over H. With $\|\cdot\|$ being the operator norm and $*$ being the adjoint operation, $\mathcal{A} = B(H)$ is a C^*-algebra.

We verified all the relevant properties before.
Note: H does not need to be finite dimensional.

Example 2 Let $\mathcal{A} = BC(\mathbb{R})$ i.e. the collection of all functions $f : \mathbb{R} \to \mathbb{C}$ for which f is bounded and continuous. This collection is clearly a vector space with a natural product. Let $*$ be complex conjugation and denote by $\|\cdot\|$ the supremum norm, i.e.

$$\|f\| = \sup_{x \in \mathbb{R}} |f(x)|.$$

In this case, $\mathcal{A} = BC(\mathbb{R})$ is a C^*-algebra.

Note: Since $fg = gf$ for all $f, g \in \mathcal{A}$, $\mathcal{A}$ is said to be a commutative C^*-algebra.

This is not the case with $B(H)$ if $\dim(H) \geq 2$.

<u>Example 3</u> Let H be a complex Hilbert space.

Let $\mathcal{C} \subset B(H)$ be the collection of compact operators on H. Then $\mathcal{C}$ is a C^*-subalgebra of $B(H)$.

<u>Note</u>: If $\dim(H) = +\infty$, then $\mathbb{1} \notin \mathcal{C}$.

<u>Note</u>: Here is a general fact:

Let H be a complex Hilbert space. Let $\mathcal{A} \subset B(H)$ be any norm closed subalgebra of $B(H)$ which is also a self-adjoint subset of $B(H)$. Then $\mathcal{A}$ is a C^* sub-algebra of $B(H)$.

- It is not hard to prove this fact.
- It is not hard to show that the compact operators satisfy these constraints.

Here is a useful fact.

Let $\mathcal{A} \neq \{0\}$ be a unital C^*-algebra.

Then
- $\mathbb{1} \in \mathcal{A}$ is unique.
- $\mathbb{1}^* = \mathbb{1}$
- $\|\mathbb{1}\| = 1$.

On positive elements

Let $\mathfrak{A}$ be a C^*-algebra.

An element $A \in \mathfrak{A}$ is said to be positive, which we denote by $A \geq 0$, if there is some $B \in \mathfrak{A}$ and

$$A = B^* B.$$

Note: We should probably say A is non-negative ...

Note: It is easy to check that: $A \geq 0 \Rightarrow A \in \mathfrak{A}_{sa}$.

It is an important fact that this notion of positivity allows us to ~~prove~~ define a partial order on $\mathfrak{A}_{sa}$.

Let $A, B \in \mathfrak{A}_{sa}$. We write that $A \geq B$ if and only if $A - B \geq 0$. (Note: we may also write $B \leq A$.)

The following proposition contains some useful facts

Proposition: Let $\mathfrak{A}$ be a unital C^*-algebra.

i) If $A \geq 0$ and $A \leq 0$, then $A = 0$.

ii) If $A \geq B$ and $B \geq C$, then $A \geq C$.

iii) If $A \geq 0$, then $\|A\| \mathbb{1} \geq A$.

iv) If $A \geq B \geq 0$, then

$$C^* A C \geq C^* B C \geq 0 \quad \text{for all } C \in \mathfrak{A}.$$