Lecture 16

On Quantum Spin Systems:

Quantum spin systems are simplified models of condensed matter physics. They are crucial in the development of quantum statistical mechanics and quantum field theory. More recently, they have become the standard models for quantum computation and quantum information theory. In words, they are models of many-body, interacting quantum systems.

Most physically relevant quantum spin models are defined on $\mathbb{Z}^\nu$ for some integer $\nu \geq 1$. As we will see, the lattice structure of $\mathbb{Z}^\nu$ is not important for the basic structure and also some basic theorems. (More on this later.)

To begin, let us consider models on $\mathbb{Z}^\nu$ where $\nu \geq 1$.

For each site $x \in \mathbb{Z}^\nu$, we associate a finite dimensional Hilbert space $\mathcal{H}_x \cong \mathbb{C}^{d_x}$. Typically, one assumes $d_x \geq 2$ to avoid trivialities.

Terminology

$\mathcal{H}_x$ is said to be

- the Hilbert space of states associated to the site $x \in \mathbb{Z}^\nu$
- the "degrees of freedom" associated to the site $x \in \mathbb{Z}^\nu$
- the "single-site" Hilbert space.

Common Example

For many models, $d_x = 2$ and so $\mathcal{H}_x \cong \mathbb{C}^2$.

A basis for this Hilbert space is $\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\}$.

It is not uncommon for:

$\begin{pmatrix}1\\0\end{pmatrix}$ - to be referred to as "spin-up".

$\begin{pmatrix}0\\1\end{pmatrix}$ - to be referred to as "spin-down".

Remark:

Let $n \geq 1$ be the dimension of the single site Hilbert space.

This dimension can be relabeled by a "spin-variable" s:

Set $n = 2s+1$.

In this case, if $n \geq 1$ (i.e. $n \in \mathbb{N}$), then $s \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \dots\}$
(i.e. $s \in \frac{1}{2}\mathbb{N}_0$)

These "spin-variables" are often used to designate specific orthonormal bases of $\mathcal{H} = \mathbb{C}^n$

dimension	spin-value	basis
$n=1$	$s=0$	$\{\|0\rangle\}$
$n=2$	$s=1/2$	$\{\|-1/2\rangle, \|1/2\rangle\}$
$n=3$	$s=1$	$\{\|-1\rangle, \|0\rangle, \|1\rangle\}$
$n=4$	$s=3/2$	$\{\|-\frac{3}{2}\rangle, \|-\frac{1}{2}\rangle, \|\frac{1}{2}\rangle, \|\frac{3}{2}\rangle\}$
$\vdots$	$\vdots$	$\vdots$
$n=n$	$s=s$	$\{\|-s\rangle, \|-s+1\rangle, \dots, \|s-1\rangle, \|s\rangle\}$

A brief comment on related quantum systems that are <u>not</u> quantum spin systems.

One could imagine associating a general Hilbert space to each site $x \in \mathbb{Z}^\nu$. For example, one might consider

$$\mathcal{H}_x = L^2(\mathbb{R}^n, dx).$$

This separable but infinite dimensional Hilbert space does <u>not</u> describe a spin at site x because it has more than finitely many degrees of freedom. Systems of this type are generally called: <u>quantum</u> <u>lattice</u> <u>systems</u>. Many of the results we will prove also hold in this context, but much more effort is required in the proofs.

Our goal is to introduce a number of important concepts in the simplest possible setting.

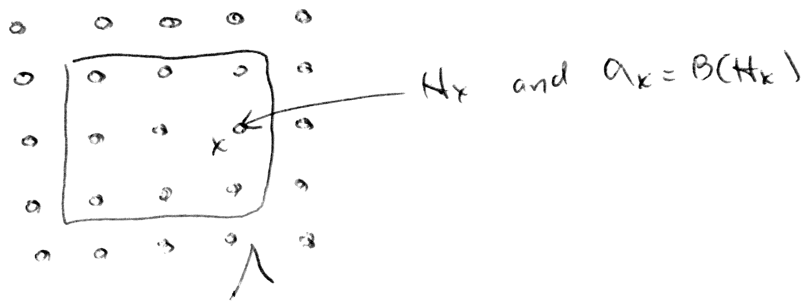
Back to quantum spin systems on $\mathbb{Z}^\nu$.

For each site $x \in \mathbb{Z}^\nu$, we have associated a finite-dimensional Hilbert space $\mathcal{H}_x$. In addition to this, we will also associate:

$\mathcal{A}_x = B(\mathcal{H}_x)$ - the collection of bounded linear operators over $\mathcal{H}_x$. As we have seen, $\mathcal{A}_x$ is a C^*-algebra. It is often referred to as the algebra of observables associated to the site $x \in \mathbb{Z}^\nu$. If $\mathcal{H}_x \cong \mathbb{C}^{d_x}$, then $\mathcal{A}_x = M_{d_x}(\mathbb{C})$.

On Systems in finite volume

Let $\Lambda \subset \mathbb{Z}^\nu$ be a finite set.



The Hilbert space of states in Λ is defined to be

$$\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x .$$

In words, this is the tensor product over all the finitely many single site Hilbert spaces $\mathcal{H}_x$ when $x \in \Lambda$.

The algebra of observables corresponding to Λ is

$$\mathcal{a}_\Lambda = \mathcal{B}(\mathcal{H}_\Lambda) \cong \bigotimes_{x \in \Lambda} \mathcal{B}(\mathcal{H}_x) = \bigotimes_{x \in \Lambda} \mathcal{a}_x .$$

<u>Note</u>: The dimension of these spaces grows very quickly. Take the simplest case where $\mathcal{H}_x = \mathbb{C}^2$ for all $x \in \mathbb{Z}^\nu$. In this case, for any finite $\Lambda \subset \mathbb{Z}^\nu$,

$$\dim(\mathcal{H}_\Lambda) = \prod_{x \in \Lambda} \dim(\mathcal{H}_x) = \prod_{x \in \Lambda} 2 = 2^{|\Lambda|}$$

where $|\Lambda|$ is the <u>cardinality of Λ</u>, i.e. the number of sites in Λ.

The algebra of observables has a dimension which grows even faster:

$$\dim(\mathfrak{A}_\Lambda) = \prod_{x\in\Lambda} \dim(\mathfrak{A}_x) = \prod_{x\in\Lambda} \dim(M_2(\mathbb{C})) = \prod_{x\in\Lambda} 2^2$$

$$= 2^{2|\Lambda|}.$$

This shows that the dimension of these finite volume systems grows exponentially in the size of the volume. From this fact, we see an early indication that, although these are "simple" models, they have a rich complexity.

<u>Note that</u>

size of Volume	dimension of states	dimension of observables
$\lvert\Lambda\rvert = 1$	$2^{\lvert\Lambda\rvert} = 2$	$2^{2\lvert\Lambda\rvert} = 4$
$\lvert\Lambda\rvert = 2$	$2^{\lvert\Lambda\rvert} = 4$	$2^{2\lvert\Lambda\rvert} = 16$
$\lvert\Lambda\rvert = 3$	$2^{\lvert\Lambda\rvert} = 8$	$2^{2\lvert\Lambda\rvert} = 64$
$\lvert\Lambda\rvert = 4$	$2^{\lvert\Lambda\rvert} = 16$	$2^{2\lvert\Lambda\rvert} = 256$
$\vdots$	$\vdots$	$\vdots$
$\lvert\Lambda\rvert = 10$	$2^{\lvert\Lambda\rvert} = 1024$	$2^{\lvert\Lambda\rvert} = 1,048,576$

Already at 10 sites, calculations become difficult for a computer.

Let's return to our common example.

<u>Example</u>: Suppose $\mathcal{H}_x = \mathbb{C}^2$ for all $x \in \mathbb{Z}^\nu$.

Recall the well-known Pauli-Matrices

$$\sigma^0 = \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We saw that $\{ \frac{1}{2}\sigma^0, \frac{1}{2}\sigma^1, \frac{1}{2}\sigma^2, \frac{1}{2}\sigma^3 \}$ form an orthonormal basis of $\mathcal{A}_x = \mathcal{B}(\mathcal{H}_x) = M_2(\mathbb{C})$.

<u>Note</u>: The canonical basis $\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \}$ of $\mathcal{H}_x = \mathbb{C}^2$ can be regarded as an orthonormal basis of eigen vectors of $\frac{1}{2}\sigma^3$. In this case, one often writes:

$|-\frac{1}{2}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ since this is the eigenvector of $\frac{1}{2}\sigma^3$ with eigenvalue $\lambda = -1/2$.

$|\frac{1}{2}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ since this is the eigenvector of $\frac{1}{2}\sigma^3$ with eigenvalue $\lambda = +1/2$.

Since $n = 2$, $s = 1/2$ and this is a labeling of an orthonormal basis of $\mathbb{C}^2$ with these "spin labels".

In any finite volume $\Lambda \subset \mathbb{Z}^\nu$, these single-site Pauli-matrices, i.e. $\{\sigma^0, \sigma^1, \sigma^2, \sigma^3\}$ as above, can be extended so that they correspond to observables on the finite volume Λ.

In fact, for each $x \in \Lambda$ and any $j \in \{0,1,2,3\}$, an observable $\sigma_x^j \in \mathcal{A}_\Lambda$ is defined by setting

$$\sigma_x^j = \mathbb{1} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \sigma^j \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$$

put σ^j in the $\underline{x\text{-th}}$ factor

(Note: Here we have chosen an enumeration of Λ ...)

σ_x^j corresponds to σ^j acting at site x in the finite volume Λ.

Terms of the form

$$\sigma_x^j \sigma_y^k \in \mathcal{A}_\Lambda$$

correspond to σ^j at site x interacting with σ^k at site y in the finite volume Λ.

We now consider more general observables of this type.

Back to general finite volume quantum spin systems.

Observables and Support

Let $\Lambda_0 \subset \Lambda \subset \mathbb{Z}^\nu$ be finite sets.

Any $A \in \mathcal{A}_{\Lambda_0}$ can be identified with

$$\tilde{A} = A \otimes \mathbb{1}_{\Lambda \setminus \Lambda_0} \in \mathcal{A}_\Lambda .$$

(Here $\Lambda \setminus \Lambda_0 = \{ y \in \Lambda : y \notin \Lambda_0 \}$)

Using this identification, we regard $\mathcal{A}_{\Lambda_0} \subset \mathcal{A}_\Lambda$. In fact, one checks that $\mathcal{A}_{\Lambda_0}$ is a sub algebra of $\mathcal{A}_\Lambda$ under this identification.

The <u>support</u> of an observable $A \in \mathcal{A}_\Lambda$ is the minimal set $X \subset \Lambda$ for which

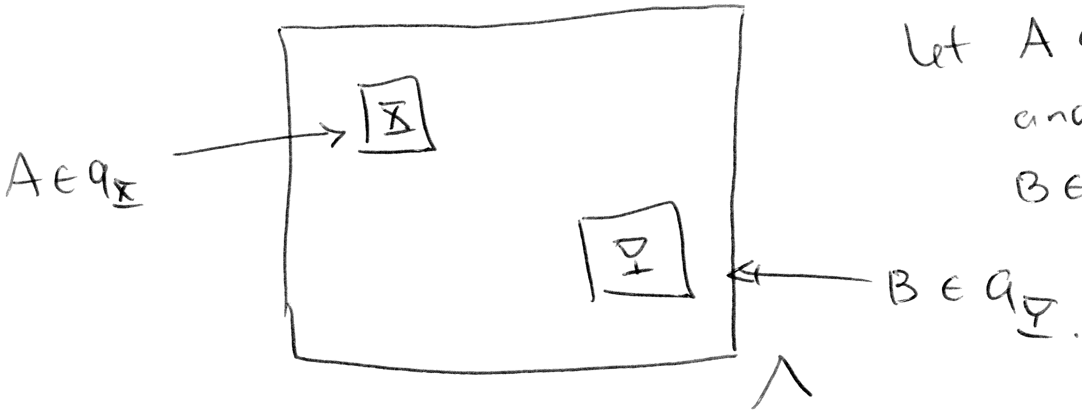
$$A = A' \otimes \mathbb{1}_{\Lambda \setminus X} \quad \text{for some } A' \in \mathcal{A}_X .$$

In this case, we write $X = \text{supp}(A)$ and call X the support of A.

In words, the support of A is the set of sites on which A acts non-trivially. (Here we think of the identity $\mathbb{1}$ as "trivial".)

Let $X, Y \subset \mathbb{Z}^\nu$ be finite disjoint sets, i.e. $X \cap Y = \phi$.

Let $\Lambda \subset \mathbb{Z}^\nu$ be any finite set with $X \cup Y \subset \Lambda$.



Let $A \in \mathfrak{A}_X$ and $B \in \mathfrak{A}_Y$.

Since $X \subset \Lambda$, A can be identified with $A \otimes 1\!\!1_{\Lambda \setminus X} \in \mathfrak{A}_\Lambda$.

Since $Y \subset \Lambda$, B can be identified with $B \otimes 1\!\!1_{\Lambda \setminus Y} \in \mathfrak{A}_\Lambda$.

In this case, one can calculate the commutator:

$$[A \otimes 1\!\!1_{\Lambda \setminus X}, B \otimes 1\!\!1_{\Lambda \setminus Y}] = 0 \quad \text{on } \mathfrak{A}_\Lambda .$$

In words, the tensor product structure of the Hilbert space (and hence the observable algebra) guarantees that observables with disjoint supports commute.

As a "short-hand", we often suppress these identities and just write: If $X \cap Y = \phi$, then

$$[A, B] = 0 \quad \text{for } A \in \mathfrak{A}_X \text{ and } B \in \mathfrak{A}_Y .$$

The above commutator is then understood to be the commutator of these observables extended to any $\mathfrak{A}_\Lambda$ where $X \cup Y \subset \Lambda$.

Interactions and Dynamics

Mathematically, models are defined through global objects called interactions. An interaction Φ on $\mathbb{Z}^\nu$ is a map from the set of finite subsets of $\mathbb{Z}^\nu$ into the observable algebras which satisfies:

For each $X \subset \mathbb{Z}^\nu$ finite

i) $\Phi(X) \in \mathcal{A}_X$ ie Φ is strictly local

ii) $\Phi(X)^* = \Phi(X)$ i.e. Φ is term-wise self-adjoint

Given an interaction Φ, to each finite set $\Lambda \subset \mathbb{Z}^\nu$ there is a corresponding finite-volume Hamiltonian given by

$$H_\Lambda^\Phi = \sum_{X \subset \Lambda} \Phi(X)$$

Some comments:

1) The sum is over all subsets of Λ.

2) Each term $\Phi(X)$ is to be regarded as an element of $\mathcal{A}_\Lambda$. One should write $\Phi(X) \otimes \mathbb{1}_{\Lambda \setminus X}$, but this becomes too cumbersome. As a result, $H_\Lambda^\Phi \in \mathcal{A}_\Lambda$.

3) Since each term $\Phi(X)$ is self-adjoint (in $\mathcal{A}_X$) it is clear that $\Phi(X) \otimes \mathbb{1}_{\Lambda \setminus X}$ is self-adjoint in $\mathcal{A}_\Lambda$ (we checked this before.)

and thus H_Λ^Φ is self-adjoint as the sum of finitely many self-adjoint terms.

Since H_Λ^Φ is self-adjoint, we may define (using functional calculus) the corresponding Heisenberg dynamics

$$\tau_t^{\Lambda,\Phi}(A) = e^{itH_\Lambda^\Phi} A \, e^{-itH_\Lambda^\Phi} \quad \text{for all } A \in \mathcal{A}_\Lambda \text{ and } t \in \mathbb{R}.$$

As we have checked, this Heisenberg dynamics, i.e. the family $\{\tau_t^{\Lambda,\Phi}\}_{t\in\mathbb{R}}$, is a one-parameter group of automorphisms of $\mathcal{A}_\Lambda$. It is this that we will regard as the relevant finite-volume dynamics for our models.

A first goal is to find conditions on Φ which guarantee the existence of a limiting dynamics as $\Lambda \to \mathbb{Z}^\nu$. Such a proof is called the existence of the dynamics in the thermodynamic limit.

First, an example.

<u>Example</u> The Quantum Heisenberg chain.

Consider $\mathbb{Z}$. (Here chain means one-dimensional.)

To each $x \in \mathbb{Z}$, set $\mathcal{H}_x = \mathbb{C}^2$.

For any $J \in \mathbb{R}$, define an interaction Φ_J^H ← parameter (H: Heisenberg)

by setting

$$\Phi_J^H(X) = \begin{cases} -J\,\vec{\sigma}_x \cdot \vec{\sigma}_{x+1} = -J\left(\sigma_x^1\sigma_{x+1}^1 + \sigma_x^2\sigma_{x+1}^2 + \sigma_x^3\sigma_{x+1}^3\right) & \text{if } X = \{x, x+1\} \\ 0 & \text{otherwise.} \end{cases}$$

In this case, for a finite volume $\Lambda = [a,b] \cap \mathbb{Z}$ one has that

$$\mathcal{H}_\Lambda = \bigotimes_{x=a}^{b} \mathbb{C}^2 = \mathbb{C}^{2^{b-a+1}}$$

$$\mathcal{A}_\Lambda = \bigotimes_{x=a}^{b} M_2(\mathbb{C}) = M_{2^{b-a+1}}(\mathbb{C})$$

and

$$H_\Lambda^{\Phi_J^H} = \sum_{X \subset \Lambda} \Phi_J^H(X) = -J \sum_{x=a}^{b-1} \vec{\sigma}_x \cdot \vec{\sigma}_{x+1}$$

- If $J > 0$, this is called the quantum ferromagnetic Heisenberg chain.
- If $J < 0$, this is called the quantum anti-ferromagnetic Heisenberg chain.