Lecture 17

On Quasi-Locality of the finite-volume dynamics in quantum spin systems

Let us recall the basic setup.

Let (Γ, d) be a countable metric space.

Often $(\Gamma, d) = (\mathbb{Z}^\nu, |\cdot|)$ where $\nu \geq 1$ and $|\cdot|$ is the Euclidean distance, but this assumption will not be necessary.

To each site $x \in \Gamma$ associate:

$\mathcal{H}_x = \mathbb{C}^{d_x}$ - a finite dimensional (complex) Hilbert space of states at $x \in \Gamma$

and

$\mathcal{A}_x = \mathcal{B}(\mathcal{H}_x) = M_{d_x}(\mathbb{C})$ - the bounded linear operators over $\mathcal{H}_x$ (which in this case is the $d_x \times d_x$ matrices with complex entries) which we refer to as the algebra of observables associated to the site $x \in \Gamma$.

For any finite $\Lambda \subset \Gamma$, consider

$$\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x \quad \text{- the Hilbert space of states in } \Lambda$$

$$\mathcal{A}_\Lambda = \mathcal{B}(\mathcal{H}_\Lambda) \cong \bigotimes_{x \in \Lambda} \mathcal{B}(\mathcal{H}_x) = \bigotimes_{x \in \Lambda} \mathcal{A}_x$$

the algebra of observables associated to Λ.

Last class we also introduced interactions.

Recall that a map Φ from the set of finite subsets of Γ to the local observable algebras satisfying:

For any finite $X \subset \Gamma$,

- $\Phi(X) \in \mathcal{A}_X$ i.e. Φ is strictly local
- $\Phi(X)^* = \Phi(X)$ i.e. Φ is termwise self adjoint

is said to be an <u>interaction</u> on (Γ, d).

Given an interaction Φ on (Γ, d), to any finite $\Lambda \subset \Gamma$ one may associate a finite volume Hamiltonian

$$H_\Lambda^\Phi = \sum_{X \subset \Lambda} \Phi(X).$$

As we discussed before, $(H_\Lambda^\Phi)^* = H_\Lambda^\Phi \in \mathcal{A}_\Lambda$ and so by functional calculus: For each $t \in \mathbb{R}$,

$$\tau_t^{\Lambda,\Phi}(A) = e^{itH_\Lambda^\Phi} A e^{-itH_\Lambda^\Phi} \quad \text{for all } A \in \mathcal{A}_\Lambda$$

is the corresponding Heisenberg dynamics in finite volume associated to Φ. The family $\{\tau_t^{\Lambda,\Phi}\}_{t \in \mathbb{R}}$ is a one parameter group of automorphisms of $\mathcal{A}_\Lambda$.

For any interaction Φ on (Γ, d) and each finite $\Lambda \subset \Gamma$, the Heisenberg dynamics $\tau_t^{\Lambda,\Phi}(A)$ is not strictly local. In fact, if you look at the support, one finds that generally

$$\mathrm{supp}(\tau_t^{\Lambda,\Phi}(A)) = \begin{cases} \mathrm{supp}(A) & \text{if } t = 0 \\ \Lambda & \text{, if } t \neq 0. \end{cases}$$

In words, the dynamics associated to these quantum systems is non-relativistic. There is no finite speed of light (or sound...).

For a "large" volume Λ and an observable A supported on a particular site in Λ, the time-evolution $\tau_t^{\Lambda,\Phi}(A)$ is, for any $t \neq 0$, immediately supported on all of Λ. From this perspective, this dynamics is <u>not local</u>.

One can understand this phenomenon as follows. Since the finite-volume Hamiltonians are bounded, one can prove that the dynamics has a well-defined Taylor expansion at $t=0$. In fact,

$$\tau_t^{\Lambda,\Phi}(A) = \sum_{n=0}^{\infty} \frac{\left.\frac{d^n}{dt^n}\left(\tau_t^{\Lambda,\Phi}(A)\right)\right|_{t=0}}{n!} t^n$$

is absolutely convergent in the norm on $\mathfrak{A}$.

Using functional calculus, the derivative of this dynamics is easy to calculate:

$$\frac{d}{dt}\tau_t^{\Lambda,\Phi}(A) = \frac{d}{dt}\left(e^{itH_\Lambda^\Phi} A e^{-itH_\Lambda^\Phi} \right)$$

$$= \frac{d}{dt}\left(e^{itH_\Lambda^\Phi}\right) A e^{-itH_\Lambda^\Phi} + e^{itH_\Lambda^\Phi} A \frac{d}{dt}\left(e^{-itH_\Lambda^\Phi}\right)$$

$$= e^{itH_\Lambda^\Phi} iH_\Lambda^\Phi A e^{-itH_\Lambda^\Phi} + e^{itH_\Lambda^\Phi} A (-iH_\Lambda^\Phi) e^{-itH_\Lambda^\Phi}$$

$$= i e^{itH_\Lambda^\Phi} \left(H_\Lambda^\Phi A - A H_\Lambda^\Phi \right) e^{-itH_\Lambda^\Phi}$$

$$= i \tau_t^{\Lambda,\Phi}\left([H_\Lambda^\Phi, A] \right).$$

and then inductively one sees that:

$$\frac{d^2}{dt^2}\tau_t^{\Lambda,\Phi}(A) = (i)^2 \tau_t^{\Lambda,\Phi}\left([H_\Lambda^\Phi, [H_\Lambda^\Phi, A]] \right)$$

$$\frac{d^3}{dt^3}\tau_t^{\Lambda,\Phi}(A) = (i)^3 \tau_t^{\Lambda,\Phi}\left([H_\Lambda^\Phi, [H_\Lambda^\Phi, [H_\Lambda^\Phi, A]]] \right)$$

$$\vdots$$

$$\frac{d^n}{dt^n}\tau_t^{\Lambda,\Phi}(A) = (i)^n \tau_t^{\Lambda,\Phi}\left([H_\Lambda^\Phi, \ldots [H_\Lambda^\Phi, A] \cdots] \right)$$

n-fold commutator with

$[H_\Lambda^\Phi, \cdot\,]$

We conclude that this Taylor series is:

$$\tau_t^{\Lambda,\Phi}(A) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} [H_\Lambda^\Phi, [H_\Lambda^\Phi, \cdots [H_\Lambda^\Phi, A] \cdots]]$$

$$= A + it[H_\Lambda^\Phi, A] + \frac{(it)^2}{2!}[H_\Lambda^\Phi, [H_\Lambda^\Phi, A]] + \cdots$$

Let's examine this further in a more concrete model.

Let $\Gamma = \mathbb{Z}$ and $d = 1 \cdot 1$.

Take a nearest neighbor interaction Φ i.e.

$$\Phi(X) = \begin{cases} h_{x,x+1} & \text{if } X = \{x, x+1\} \\ 0 & \text{otherwise.} \end{cases}$$

For any $N \geq 1$, let $\Lambda_N = [-N, N] \cap \mathbb{Z}$.

In this case,

$$H_{\Lambda_N}^\Phi = \sum_{X \subset \Lambda_N} \Phi(X) = \sum_{x=-N}^{N-1} h_{x,x+1}$$

Let $A \in \mathcal{A}_{\{0\}}$.

Then

$$[H_{\Lambda_N}^\Phi, A] = \sum_{x=-N}^{N-1} [h_{x,x+1}, A] = [h_{-1,0}, A] + [h_{0,1}, A]$$

as all other Hamiltonian terms have support which is disjoint from A.

We conclude:

$$\operatorname{supp}([H_{\Lambda_N}^\Phi, A]) = \{-1, 0, 1\}.$$

We similarly have that

$$
\begin{aligned}
[H_\Lambda^\Phi, [H_\Lambda^\Phi, A]] &= [H_\Lambda^\Phi, [h_{-1,0}, A]] + [H_\Lambda^\Phi, [h_{0,1}, A]] \\
&= \sum_{x=-N}^{N-1} [h_{x,x+1}, [h_{-1,0}, A]] + \sum_{x=-N}^{N-1} [h_{x,x+1}, [h_{0,1}, A]] \\
&= [h_{-2,-1}, [h_{-1,0}, A]] + [h_{-1,0}, [h_{-1,0}, A]] + [h_{0,1}, [h_{-1,0}, A]] \\
&\quad + [h_{-1,0}, [h_{0,1}, A]] + [h_{0,1}, [h_{0,1}, A]] + [h_{1,2}, [h_{0,1}, A]]
\end{aligned}
$$

and so

$$
\operatorname{supp}([H_\Lambda^\Phi, [H_\Lambda^\Phi, A]]) = \{-2,-1,0,1,2\}.
$$

One checks that each additional commutation with this nearest neighbor Hamiltonian expands the support one unit to the left and one unit to the right.

In this case, one concludes that

$$
\begin{aligned}
\tau_t^{\Lambda,\Phi}(A) &= \sum_{n=0}^{\infty} \frac{(it)^n}{n!} [H_\Lambda^\Phi, \cdots [H_\Lambda^\Phi, A] \cdots] \\
&= A + it[H_\Lambda^\Phi, A] + \frac{(it)^2}{2!} [H_\Lambda^\Phi, [H_\Lambda^\Phi, A]] + \cdots
\end{aligned}
$$

satisfies

$$
\operatorname{supp}(\tau_t^{\Lambda,\Phi}(A)) = \begin{cases} \operatorname{supp}(A) & \text{if } t = 0 \\ \Lambda & \text{if } t \neq 0 \end{cases}
$$

as we claimed before.

It turns out that although these finite volume Heisenberg dynamics are <u>not</u> local, they do satisfy a property called <u>quasi-locality</u>. The 1st estimates of this type were proven by Lieb and Robinson in 1972. In recent years, these bounds have been generalized in many directions and several new applications have been found. We will now turn to a precise statement of these bounds.

Before we do so, we need to introduce two important notions.

First, we need a "regularity condition" on the countable metric space (Γ, d). We will express this regularity in terms of the existence of a function with some "nice" properties.

Next, we need to restrict our attention to a class of interactions that satisfy an appropriate decay assumption. We will express this decay in terms of the function introduced above.

A regularity condition on (Γ, d).

We will say that $F : [0,\infty) \to (0,\infty)$ that is non-increasing is an F-function on (Γ, d) if:

i) F is uniformly summable on Γ, i.e.

$$\|F\| = \sup_{x \in \Gamma} \sum_{y \in \Gamma} F(d(x,y)) < \infty .$$

Note: F positive and summable means that Γ is, at most, countable!

and

ii) F satisfies a convolution condition, i.e. there is a number $C_F \geq 0$ for which: given any $x, y \in \Gamma$

$$\sum_{z \in \Gamma} F(d(x,z)) \cdot F(d(z,y)) \leq C_F \cdot F(d(x,y)) .$$

We will say that (Γ, d) is regular if there is an F-function on (Γ, d).

Example Let $\nu \geq 1$ be an integer. Let $\Gamma = \mathbb{Z}^\nu$ and take $d = |\cdot|$ the Euclidean metric. We claim that $F : [0,\infty) \to (0,\infty)$ given by

$$F(r) = \frac{1}{(1+r)^{\nu+1}} \quad \text{for all } r \geq 0$$

is an F-function on $(\mathbb{Z}^\nu, |\cdot|)$.

First, it is clear that $F(r) > 0$ for all $r \geq 0$.

It is also clear that F is non-increasing.

Check uniform summability:

Let $x \in \mathbb{Z}^\nu$. Then

$$\sum_{y \in \mathbb{Z}^\nu} F(d(x,y)) = \sum_{y \in \mathbb{Z}^\nu} \frac{1}{(1+|x-y|)^{\nu+1}}$$

change variables with $z = x-y$

$$= \sum_{z \in \mathbb{Z}^\nu} \frac{1}{(1+|z|)^{\nu+1}}$$

It is clear from calculus that the final quantity above is finite. It is also independent of x. Thus

$$\|F\| = \sup_{x \in \mathbb{Z}^\nu} \sum_{y \in \mathbb{Z}^\nu} F(d(x,y)) = \sum_{z \in \mathbb{Z}^\nu} \frac{1}{(1+|z|)^{\nu+1}} < \infty .$$

Check the convolution condition:

Note that: For any $a \geq 0$, $b \geq 0$, and $p \geq 0$, the simple estimate

$$(a+b)^p \leq (2 \cdot \max\{a,b\})^p = 2^p \cdot \max\{a,b\}^p$$
$$\leq 2^p (a^p + b^p)$$

is always true.

In this case, if $x, y, z \in \mathbb{Z}^\nu$, then

$$1 + |x-y| \le 2 + |x-z| + |z-y| = (1+|x-z|) + (1+|z-y|)$$

$$\Rightarrow (1+|x-y|)^{\nu+1} \le \big((1+|x-z|) + (1+|z-y|)\big)^{\nu+1} \le 2^{\nu+1}\left((1+|x-z|)^{\nu+1} + (1+|z-y|)^{\nu+1}\right) \quad (*)$$

Now let's use this to verify the convolution condition.

Fix any $x, y \in \mathbb{Z}^\nu$.

Consider

$$\sum_{z \in \mathbb{Z}^\nu} \frac{F(d(x,z)) \cdot F(d(z,y))}{F(d(x,y))}$$

we will find an upper bound on this independent of x and y.

$$= \sum_{z \in \mathbb{Z}^\nu} \frac{(1+|x-y|)^{\nu+1}}{(1+|x-z|)^{\nu+1}(1+|z-y|)^{\nu+1}}$$

$$\overset{(*)}{\le} \sum_{z \in \mathbb{Z}^\nu} \frac{2^{\nu+1}(1+|x-z|)^{\nu+1} + 2^{\nu+1}(1+|z-y|)^{\nu+1}}{(1+|x-z|)^{\nu+1}(1+|z-y|)^{\nu+1}}$$

$$= 2^{\nu+1} \sum_{z \in \mathbb{Z}^\nu} \frac{1}{(1+|z-y|)^{\nu+1}} + 2^{\nu+1} \sum_{z \in \mathbb{Z}^\nu} \frac{1}{(1+|x-z|)^{\nu+1}}$$

$$= 2^{\nu+2} \sum_{z \in \mathbb{Z}^\nu} \frac{1}{(1+|z|)^{\nu+1}} = 2^{\nu+2} \|F\|. < \infty$$

This shows that $(\mathbb{Z}^\nu, |\cdot|)$ is regular in the sense that there is an F-function on $(\mathbb{Z}^\nu, |\cdot|)$.

Here is an interesting general fact.

Suppose F_0 is an F-function on (Γ, d).

We claim that: For any $a > 0$, the function

$$F_a(r) = e^{-ar} \cdot F_0(r) \qquad \text{for } r \geq 0$$

is also an F-function on (Γ, d)

<u>Note</u>: For each $a > 0$, F_a is positive an non-increasing.

<u>Note that</u>

$$\|F_a\| = \sup_{x \in \Gamma} \sum_{y \in \Gamma} F_a(d(x,y))$$

$$= \sup_{x \in \Gamma} \sum_{y \in \Gamma} e^{-ar} F_0(d(x,y))$$

use that $e^{-ar} \leq 1$ for all $r \geq 0$.

$$\leq \sup_{x \in \Gamma} \sum_{y \in \Gamma} F_0(d(x,y))$$

$$= \|F_0\|$$

It is also easy to see that:

If $x, y, z \in \Gamma$ then.

$$d(x,y) \le d(x,z) + d(z,y)$$ by the triangle inequality for the metric d.

Thus for any $a > 0$, since e^{-ar} is a decreasing function, we have that

$$e^{-ad(x,y)} \ge e^{-a(d(x,z)+d(z,y))} = e^{-ad(x,z)} \cdot e^{-ad(z,y)}$$

Thus

$$\sum_{z \in \Gamma} F_a(d(x,z))\, F_a(d(z,y))$$

$$= \sum_{z \in \Gamma} e^{-ad(x,z)} \cdot e^{-ad(z,y)} \cdot F_0(d(x,z)) \cdot F_0(d(z,y))$$

$$\le \sum_{z \in \Gamma} e^{-ad(x,y)} \cdot F_0(d(z,x)) \cdot F_0(d(z,y))$$

$$\le e^{-ad(x,y)} \cdot C_{F_0}\, F_0(d(x,y))$$

$$= C_{F_0}\, F_a(d(x,y))$$

Thus $C_{F_a} \le C_{F_0}$ also holds.