Lecture 20

Lieb-Robinson Bounds

Recall: Let (Γ, d) be a regular metric space and let F be an F-function on (Γ, d). We say that an interaction $\Phi \in \mathcal{B}_F(\Gamma)$ if

$$\|\Phi\|_F = \sup_{x,y \in \Gamma} \sum_{\substack{X \subset \Gamma: \\ x,y \in X}} \frac{\|\Phi(X)\|}{F(d(x,y))} < \infty .$$

Said differently if $\Phi \in \mathcal{B}_F(\Gamma)$, then for each pair $x, y \in \Gamma$

$$\sum_{\substack{X \subset \Gamma: \\ x,y \in X}} \|\Phi(X)\| \leq \|\Phi\|_F \cdot F(d(x,y))$$

Theorem (Lieb-Robinson Bound) Let (Γ, d) be a regular metric space and $\Phi \in \mathcal{B}_F(\Gamma)$ for some F-function F on (Γ, d). Let $X, Y \subset \Gamma$ be finite disjoint sets. Let $\Lambda \subset \Gamma$ be finite and satisfy $X \cup Y \subset \Lambda$. Then, for any $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$, the bound

$$\|[\tau_t^{\Lambda,\Phi}(A), B]\| \leq \frac{2\|A\|\|B\|}{C_F}\left(e^{2\|\Phi\|_F C_F |t|} - 1\right) \sum_{x \in X} \sum_{y \in Y} F(d(x,y))$$

holds for all $t \in \mathbb{R}$.

Recall also

Last class we proved a norm-preservation lemma.

Lemma (Norm-Preservation). Let $I \subset \mathbb{R}$ be an open interval and $\mathcal{H}$ be a complex Hilbert space. Let $A, B : I \to B(\mathcal{H})$ satisfy

i) $A(t)^* = A(t)$ for all $t \in I$.

ii) $t \mapsto A(t)$ and $t \mapsto B(t)$ are norm continuous.

Then, the unique solution of the $B(\mathcal{H})$-valued initial value problem:

$$\frac{d}{dt} y(t) = -i\,[A(t), y(t)] + B(t) \quad \text{for } t \in I \quad \text{with } y(t_0) = y_0 \in B(\mathcal{H})$$
for some $t_0 \in I$

satisfies the norm bound

$$\|y(t)\| \le \|y_0\| + \int_{\min(t_0,t)}^{\max(t_0,t)} \|B(s)\|\, ds \qquad \text{for all } t \in I.$$

Lastly, the following identity will be useful.
Let $\mathcal{H}$ be a Hilbert space and $A, B, C \in B(\mathcal{H})$. Then

$$[[A,B],C] + [[B,C],A] + [[C,A],B] = 0$$

This is called the Jacobi Identity. It is easy to check by just expanding the commutators. In words, if you sum this iterated commutator on cyclic permutations of (A,B,C), then you get 0.

Proof of the Lieb-Robinson Bound

Fix finite disjoint sets $X, Y \subset \Gamma$. Let $\Lambda \subset \Gamma$ be finite with $X \cup Y \subset \Lambda$ and take $A \in \mathfrak{A}_X$ and $B \in \mathfrak{A}_Y$.

Define a function $f : \mathbb{R} \to B(\mathcal{H}_\Lambda) = \mathfrak{A}_\Lambda$ by setting

$$f(t) = [\tau_t^{\Lambda,\Phi}(A), B].$$

Our goal is to estimate the norm of this function. To do so, let's 1st calculate it's derivative.

$$f'(t) = [\,i\,\tau_t^{\Lambda,\Phi}([H_\Lambda^\Phi, A]), B\,]$$

$$= i\,[\,\tau_t^{\Lambda,\Phi}(\,[\sum_{\substack{Z\subset\Lambda:\\ Z\cap X\neq\emptyset}} \Phi(Z), A\,]), B\,]$$

use that $\text{supp}(A) = X$

$$= i \sum_{\substack{Z\subset\Lambda:\\ Z\cap X\neq\emptyset}} [\,[\tau_t^{\Lambda,\Phi}(\Phi(Z)), \tau_t^{\Lambda,\Phi}(A)], B\,]$$

use linearity

Now apply Jacobi with $\hat{A} = \tau_t^{\Lambda,\Phi}(\Phi(Z))$, $\hat{B} = \tau_t^{\Lambda,\Phi}(A)$, and $\hat{C} = B$

$$\Rightarrow f'(t) = -i \sum_{\substack{Z\subset\Lambda:\\ Z\cap X\neq\emptyset}} [\,[\tau_t^{\Lambda,\Phi}(A), B], \tau_t^{\Lambda,\Phi}(\Phi(Z))\,]$$

$$-i \sum_{\substack{Z\subset\Lambda:\\ Z\cap X\neq\emptyset}} [\,[B, \tau_t^{\Lambda,\Phi}(\Phi(Z))], \tau_t^{\Lambda,\Phi}(A)\,]$$

This can be re-written as:

$$f'(t) = -i\,[A(t), f(t)] + B(t)$$

with

$$A(t) = -\sum_{\substack{Z \subset \Lambda: \\ Z \cap X \neq \phi}} \tau_t^{\Lambda,\Phi}(\Phi(Z))$$

and

$$B(t) = -i \sum_{\substack{Z \subset \Lambda: \\ Z \cap X \neq \phi}} \left[\tau_t^{\Lambda,\Phi}(A), \left[\tau_t^{\Lambda,\Phi}(\Phi(Z)), B \right] \right]$$

One readily checks that:

i) $A(t)^* = A(t)$ for all $t \in \mathbb{R}$

and moreover

ii) $t \mapsto A(t)$ and $t \mapsto B(t)$ are both norm continuous.

In this case, the norm preservation lemma applies. Take $t_0 = 0$ and note that

$$\|f(t)\| \leq \|f(0)\| + \int_0^t \|B(s)\|\,ds \qquad \text{holds for all } t \geq 0.$$

A similar bound holds for $t < 0$ ...

Note further that:

$$\|B(t)\| \leq \sum_{\substack{Z \subset \Lambda: \\ Z \cap X \neq \phi}} \left\| \left[\tau_t^{\Lambda,\Phi}(A), \left[\tau_t^{\Lambda,\Phi}(\Phi(Z)), B \right] \right] \right\| \leq 2\|A\| \cdot \sum_{\substack{Z \subset \Lambda: \\ Z \cap X \neq \phi}} \left\| \left[\tau_t^{\Lambda,\Phi}(\Phi(Z)), B \right] \right\|$$

In this case, the norm-preservation lemma gives us the first estimate:

$$\| [\tau_t^{\Lambda,\Phi}(A), B] \| \leq \| [A,B] \| + 2\|A\| \sum_{\substack{Z \subset \Lambda: \\ Z \cap X \neq \phi}} \int_0^t \| [\tau_s^{\Lambda,\Phi}(\Phi(Z)), B] \| \, ds$$

Note that the left-hand-side and the right-hand-side look similar.
We would like to iterate this estimate.
To do so, it is convenient to define a quantity which focuses on support of an observable and time. Define

$$C_B^{\Lambda}(Z;t) = \sup_{\substack{A \in \mathcal{A}_Z: \\ A \neq 0}} \frac{\| [\tau_t^{\Lambda,\Phi}(A), B] \|}{\|A\|}$$

Note that

$$C_B^{\Lambda}(Z;t) \leq 2\|B\| \qquad \text{for all } Z \subset \Lambda \text{ and } t \in \mathbb{R}$$

Moreover

$$C_B^{\Lambda}(Z;0) \leq 2\|B\| \, \delta_{\Upsilon}(Z) \qquad \text{for all } Z \subset \Lambda \text{ and } t \in \mathbb{R}$$

where we have set

$$\delta_{\Upsilon}(Z) = \begin{cases} 1 & \text{if } Z \cap \Upsilon \neq \phi \\ 0 & \text{otherwise.} \end{cases}$$

Rewriting the 1st estimate interms of this new quantity yields

$$C_B^\Lambda(X;t) \le C_B^\Lambda(X;0) + 2 \sum_{\substack{Z\subset\Lambda:\\ Z\cap X\neq\emptyset}} \|\Phi(Z)\| \int_0^t C_B(Z;s)\,ds$$

$$\le 2\|B\|\,\delta_Y(X) + 2 \sum_{\substack{Z\subset\Lambda:\\ Z\cap X\neq\emptyset}} \|\Phi(Z)\| \int_0^t C_B(Z;s)\,ds$$

Now iterate:

$$\le 2\|B\|\,\delta_Y(X) + 2 \sum_{\substack{Z_1\subset\Lambda:\\ Z_1\cap X\neq\emptyset}} \|\Phi(Z_1)\| \int_0^t \left[2\|B\|\,\delta_Y(Z_1) + 2 \sum_{\substack{Z_2\subset\Lambda:\\ Z_2\cap Z_1\neq\emptyset}} \|\Phi(Z_2)\| \int_0^{s_1} C_B(Z_2;s_2)\,ds_2 \right] ds_1$$

$$\le 2\|B\|\,\delta_Y(X) + 2\|B\|\cdot 2 \sum_{\substack{Z_1\subset\Lambda:\\ Z_1\cap X\neq\emptyset}} \|\Phi(Z_1)\|\,\delta_Y(Z_1)\cdot \int_0^t 1\,ds_1$$

$$+ 2 \sum_{\substack{Z_1\subset\Lambda:\\ Z_1\cap X\neq\emptyset}} \|\Phi(Z_1)\| \cdot 2 \sum_{\substack{Z_2\subset\Lambda:\\ Z_2\cap Z_1\neq\emptyset}} \|\Phi(Z_2)\| \cdot \int_0^t \int_0^{s_1} \left[2\|B\|\,\delta_Y(Z_2) + 2 \sum_{\substack{Z_3\subset\Lambda:\\ Z_3\cap Z_2\neq\emptyset}} \|\Phi(Z_3)\| \int_0^{s_2} C_B(Z_3;s_3)\,ds_3 \right] ds_2\,ds_1$$

iterate again

$\le \cdots$ iterating $N \ge 3$ times produces:

$$C_B^\Lambda(X,t) \le 2\|B\| \left[\delta_Y(X) + \sum_{n=1}^{N} a_n \frac{(2t)^n}{n!} \right] + R_{N+1}(t)$$

Keep track of 2 and integration over simplex.

remainder term

For each $n \geq 1$, we have found that:

$$a_n = \sum_{\substack{Z_1 \subset \Lambda: \\ Z_1 \cap X \neq \phi}} \sum_{\substack{Z_2 \subset \Lambda: \\ Z_2 \cap Z_1 \neq \phi}} \cdots \sum_{\substack{Z_n \subset \Lambda: \\ Z_n \cap Z_{n-1} \neq \phi}} \left(\prod_{j=1}^{n} \|\Phi(Z_j)\| \right) \delta_Y(Z_n)$$

and moreover, the remainder term is given by

$$R_{N+1}(t) = 2^{N+1} \sum_{\substack{Z_1 \subset \Lambda: \\ Z_1 \cap X \neq \phi}} \sum_{\substack{Z_2 \subset \Lambda: \\ Z_2 \cap Z_1 \neq \phi}} \cdots \sum_{\substack{Z_{N+1} \subset \Lambda: \\ Z_{N+1} \cap Z_N \neq \phi}} \left(\prod_{j=1}^{N+1} \|\Phi(Z_j)\| \right) \times$$

$$\times \int_0^t \int_0^{s_1} \cdots \int_0^{s_N} \hat{C}_B(Z_{N+1}, s_{N+1}) \, ds_{N+1} \, ds_N \cdots ds_1$$

To complete the proof, we 1^{st} demonstrate that:

$R_{N+1}(t) \to 0$ as $N \to \infty$.

This will be true uniformly for t in compact subsets of $\mathbb{R}$.

Once this is proven we have established that:

$$C_B(X,t) \leq 2\|B\| \left[\delta_Y(X) + \sum_{n=1}^{\infty} a_n \frac{(2t)^n}{n!} \right] \quad \text{for all } \cancel{t \in \mathbb{R}} \; t \geq 0.$$

The claimed bound will follow from an estimate of a_n for all $n \geq 1$.

The estimate for $R_{N+1}(t)$ and a_n are similar.

We start with the remainder.

Recall the apriori estimate:

$$C_B^{\Lambda}(Z,t) \leq 2\|B\| \quad \text{for all} \quad Z \subset \Lambda \quad \text{and} \quad t \in \mathbb{R}.$$

In this case, for any $N \geq 1$

$$\int_0^t \int_0^{s_1} \cdots \int_0^{s_N} C_B^{\Lambda}(Z_{N+1}, s_{N+1})\, ds_{N+1} \cdots ds_1$$

$$\leq 2 \cdot \|B\| \cdot \int_0^t \int_0^{s_1} \cdots \int_0^{s_N} 1 \cdot ds_{N+1} \cdots ds_1$$

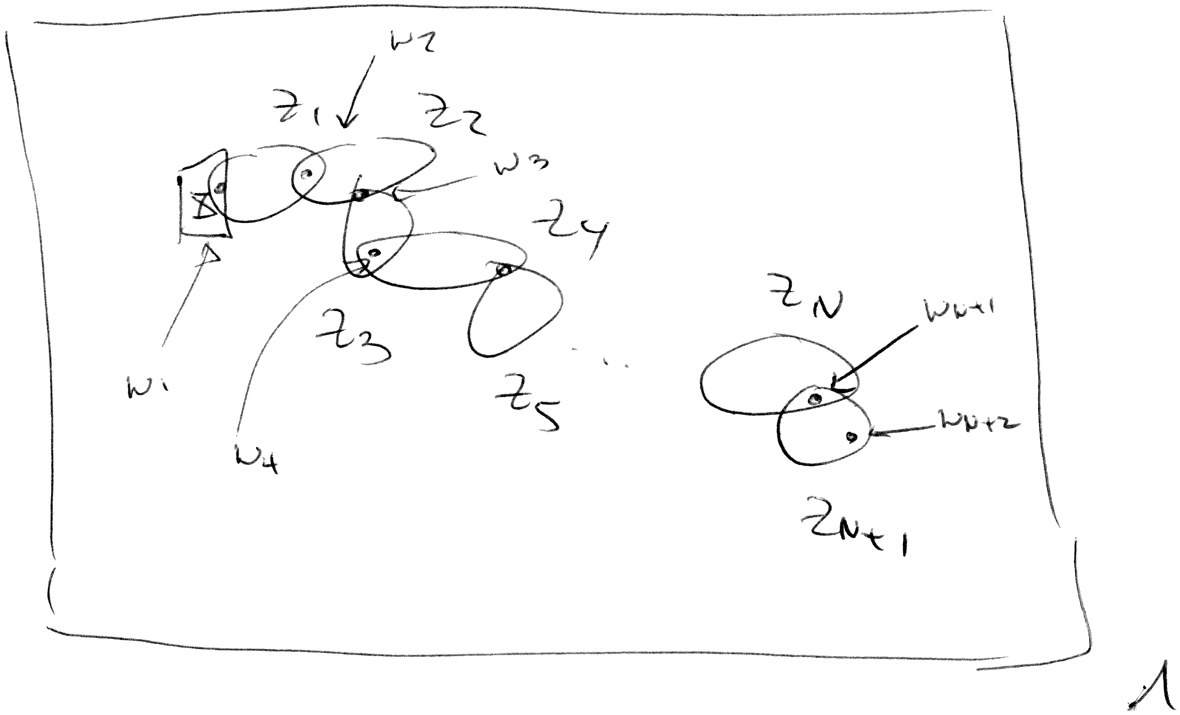
$$= 2 \cdot \|B\| \frac{\left(\int_0^t 1 \right)^{N+1}}{(N+1)!} = 2 \cdot \|B\| \cdot \frac{t^{N+1}}{(N+1)!}$$

Thus the remainder can be estimated by

$$R_{N+1}(t) \leq 2 \cdot \|B\| \cdot \frac{(2t)^{N+1}}{(N+1)!} \cdot \sum_{\substack{Z_1 \subset \Lambda: \\ Z_1 \cap X \neq \emptyset}} \sum_{\substack{Z_2 \subset \Lambda: \\ Z_2 \cap Z_1 \neq \emptyset}} \cdots \sum_{\substack{Z_{N+1} \subset \Lambda: \\ Z_{N+1} \cap Z_N \neq \emptyset}} \left(\prod_{j=1}^{N+1} \|\Phi(Z_j)\| \right)$$

To estimate this sum, we 1st recognize it as a sum over "chains of sets" which emanate from X.

One can visualize this sum as follows:



The fact that the sets intersect means that:

There is some:

$$w_1 \in Z_1 \cap X$$
$$w_2 \in Z_2 \cap Z_1$$
$$w_3 \in Z_3 \cap Z_2$$
$$\vdots$$
$$w_{N+1} \in Z_{N+1} \cap Z_N$$

For convenience, we also include a notation for some

$$w_{N+2} \in \Lambda \cap Z_{N+1}$$

By over counting sets, it is clear that:

$$\sum_{\substack{Z_1 \subset \Lambda: \\ Z_1 \cap X \neq \phi}} \sum_{\substack{Z_2 \subset \Lambda: \\ Z_2 \cap Z_1 \neq \phi}} \cdots \sum_{\substack{Z_{N+1} \subset \Lambda: \\ Z_{N+1} \cap Z_N \neq \phi}} * \leq$$

$$\sum_{w_1 \in X} \sum_{w_2 \in \Lambda} \sum_{w_3 \in \Lambda} \cdots \sum_{w_{N+1} \in \Lambda} \sum_{w_{N+2} \in \Lambda} \sum_{\substack{Z_1 \subset \Lambda: \\ w_1, w_2 \in Z_1}} \sum_{\substack{Z_2 \subset \Lambda: \\ w_2, w_3 \in Z_2}} \cdots \sum_{\substack{Z_{N+1} \subset \Lambda: \\ w_{N+1}, w_{N+2} \in Z_{N+1}}} *$$

for any non-negative quantity $*$.

For the remainder, the sum of interest is:

$$\sum_{\substack{Z_1 \subset \Lambda: \\ Z_1 \cap X \neq \phi}} \sum_{\substack{Z_2 \subset \Lambda: \\ Z_2 \cap Z_1 \neq \phi}} \cdots \sum_{\substack{Z_{N+1} \subset \Lambda: \\ Z_{N+1} \cap Z_N \neq \phi}} \left(\prod_{j=1}^{N+1} \|\Phi(Z_j)\| \right)$$

$$\leq \sum_{w_1 \in X} \sum_{w_2 \in \Lambda} \cdots \sum_{w_{N+2} \in \Lambda} \sum_{\substack{Z_1 \subset \Lambda: \\ w_1, w_2 \in Z_1}} \cdots \sum_{\substack{Z_{N+1} \subset \Lambda: \\ w_{N+1}, w_{N+2} \in Z_{N+1}}} \left(\prod_{j=1}^{N+1} \|\Phi(Z_j)\| \right)$$

Note that the final sum is:

$$\sum_{\substack{Z_{N+1} \subset \Lambda: \\ w_{N+1}, w_{N+2} \in Z_{N+1}}} \|\Phi(Z_{N+1})\| \leq \|\Phi\|_F \cdot F(d(w_{N+1}, w_{N+2}))$$

In fact, this is true for all:

$$\sum_{\substack{Z_k \subset \Lambda: \\ w_k, w_{k+1} \in Z_k}} \|\Phi(Z_k)\| \leq \|\Phi\|_F \cdot F(d(w_k, w_{k+1}))$$

$$\Rightarrow R_{N+1}(t) \leq 2 \cdot \|B\| \frac{(2t)^{N+1}}{(N+1)!} \sum_{\substack{Z_1 \subset \Lambda: \\ Z_1 \cap X \neq \phi}} \sum_{\substack{Z_2 \subset \Lambda: \\ Z_2 \cap Z_1 \neq \phi}} \cdots \sum_{\substack{Z_{N+1} \subset \Lambda: \\ Z_{N+1} \cap Z_N \neq \phi}} \left(\prod_{j=1}^{N+1} \|\Phi(Z_j)\| \right)$$

$$\leq 2 \cdot \|B\| \frac{(2\|\Phi\|_F t)^{N+1}}{(N+1)!} \sum_{w_1 \in X} \sum_{w_2 \in \Lambda} \cdots \sum_{w_{N+2} \in \Lambda} \prod_{k=1}^{N+1} F(d(w_k, w_{k+1}))$$

Now we use the property of the F-function.

Note that

$$\sum_{\omega_{N+1}\in\Lambda} F(d(\omega_N,\omega_{N+1}))\cdot F(d(\omega_{N+1},\omega_{N+2})) \le C_F \cdot F(d(\omega_N,\omega_{N+2}))$$

We can apply this for all k with $2 \le k \le N+1$.

We find that

$$R_{N+1}(t) \le 2\cdot\|B\|\cdot \frac{(2\|\Phi\|_F C_F t)^{N+1}}{(N+1)!\cdot C_F}\cdot \sum_{\omega_1\in X} \underbrace{\sum_{\omega_{N+2}\in\Lambda} F(d(\omega_1,\omega_{N+2}))}_{\text{Sum this out}}$$

$$\le \frac{2\cdot\|B\|\cdot\|F\|}{C_F}\,\frac{(2\|\Phi\|_F C_F t)^{N+1}}{(N+1)!}\cdot \underbrace{\sum_{\omega_1\in X} 1}_{|X| \leftarrow \text{The cardinality of } X}$$

$$\le \frac{2\cdot\|B\|\cdot\|F\|\cdot|X|}{C_F}\cdot\frac{(2\|\Phi\|_F C_F t)^{N+1}}{(N+1)!}$$

Thus $R_{N+1}(t) \to 0$ as $N\to\infty$ and this is uniform for t in compact subsets of $\mathbb{R}$.

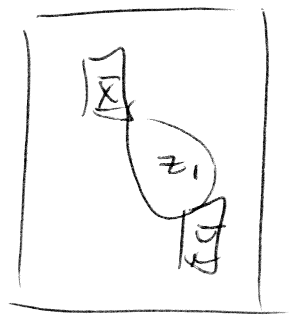
We need only estimate the remaining series.

Since we have proven that the remainder term goes to zero, we have proven that

$$C_B(X;t) \le 2\|B\| \left[\delta_Y(X) + \sum_{n=1}^{\infty} a_n \frac{(2t)^n}{n!} \right] \quad \text{for all } \cancel{t\in\mathbb{R}} \;\; t \ge 0 .$$

Let us now estimate the coefficients a_n for $n \ge 1$.

<u>Note:</u>

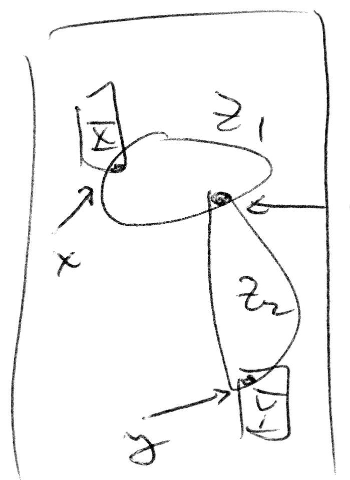


$$a_1 = \sum_{\substack{Z_1 \subset \Lambda: \\ Z_1 \cap X \ne \emptyset}} \|\Phi(Z_1)\| \, \delta_Y(Z_1)$$

$$\le \sum_{x \in X} \sum_{y \in Y} \sum_{\substack{Z \subset \Lambda: \\ x,y \in Z}} \|\Phi(Z)\|$$

$$\le \|\Phi\|_F \sum_{x \in X} \sum_{y \in Y} F(d(x,y))$$

Similarly



$$a_2 = \sum_{\substack{Z_1 \subset \Lambda: \\ Z_1 \cap X \ne \emptyset}} \sum_{\substack{Z_2 \subset \Lambda: \\ Z_2 \cap Z_1 \ne \emptyset}} \|\Phi(Z_2)\| \cdot \|\Phi(Z_1)\| \, \delta_Y(Z_2)$$

$$\le \sum_{x \in X} \sum_{y \in Y} \sum_{w \in \Lambda} \sum_{\substack{Z_1 \subset \Lambda: \\ x,w \in Z_1}} \|\Phi(Z_1)\| \sum_{\substack{Z_2 \subset \Lambda: \\ w,y \in Z_2}} \|\Phi(Z_2)\|$$

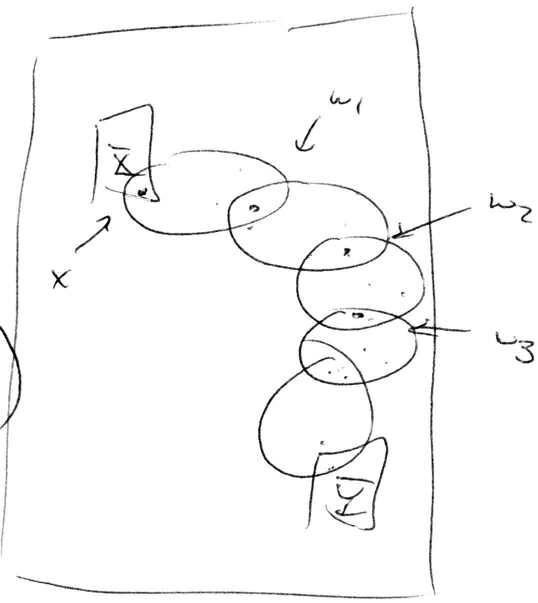
$$\le \|\Phi\|_F^2 \sum_{x \in X} \sum_{y \in Y} \sum_{w \in \Lambda} F(d(x,w)) F(d(w,y))$$

$$\le C_F \cdot \|\Phi\|_F^2 \sum_{x \in X} \sum_{y \in Y} F(d(x,y))$$

Now, $n \geq 2$ we have that

$$a_n = \sum_{\substack{Z_1 \subset \Lambda: \\ Z_1 \cap X \neq \emptyset}} \sum_{\substack{Z_2 \subset \Lambda: \\ Z_2 \cap Z_1 \neq \emptyset}} \cdots \sum_{\substack{Z_n \subset \Lambda: \\ Z_n \cap Z_{n-1} \neq \emptyset}} \left(\prod_{j=1}^{n} \|\Phi(Z_j)\| \right) \delta_Y(Z_n)$$

$$\leq \sum_{x \in X} \sum_{y \in Y} \sum_{w_1 \in \Lambda} \sum_{w_2 \in \Lambda} \cdots \sum_{w_{n-1} \in \Lambda} \cdot \sum_{\substack{Z_1 \subset \Lambda: \\ x, w_1 \in Z_1}} \sum_{\substack{Z_2 \subset \Lambda: \\ w_1, w_2 \in Z_2}} \cdots \sum_{\substack{Z_n \subset \Lambda: \\ w_{n-1}, y \in Z_n}} \left(\prod_{j=1}^{n} \|\Phi(Z_j)\| \right)$$



$$\leq \|\Phi\|_F^n \sum_{x \in X} \sum_{y \in Y} \sum_{w_1 \in \Lambda} \sum_{w_2 \in \Lambda} \cdots \sum_{w_{n-1} \in \Lambda} F(d(x, w_1))\, F(d(w_1, w_2)) \cdots F(d(w_{n-1}, y))$$

$$\leq C_F^{n-1} \|\Phi\|_F^n \sum_{x \in X} \sum_{y \in Y} F(d(x,y)))$$

Thus

$$C_B(X,t) \leq 2\|B\| \left[\delta_Y(X) + \sum_{n=1}^{\infty} a_n \frac{(2t)^n}{n!} \right]$$

$$\leq 2\|B\| \left[\delta_Y(X) + \sum_{n=1}^{\infty} C_F^{n-1} \|\Phi\|_F^n \sum_{x \in X} \sum_{y \in Y} F(d(x,y)) \frac{(2t)^n}{n!} \right]$$

$$= 2\|B\| \left[\frac{1}{C_F} \left(e^{2C_F \|\Phi\|_F t} - 1 \right) \right] \sum_{x \in X} \sum_{y \in Y} F(d(x,y)).$$

This proves the Theorem for $t \geq 0$. $t < 0$ is similar.