Lecture 21

On Infinite Systems and the Existence of the Dynamics in the Thermodynamic limit.

On Infinite Systems:

Let (Γ, d) be a regular metric space.

To each $x \in \Gamma$ associate:

$$H_x = \mathbb{C}^{d_x} \quad \text{for some } d_x \geq 2 \qquad \text{(a Hilbert space of states)}$$

and

$$\mathcal{A}_x = \mathcal{B}(H_x) = M_{d_x}(\mathbb{C}) \qquad \text{(the single site observables)}$$

For any $\Lambda \subset \Gamma$ finite, set

$$H_\Lambda = \bigotimes_{x \in \Lambda} H_x \qquad \text{(the finite volume Hilbert space of states)}$$

and

$$\mathcal{A}_\Lambda = \mathcal{B}(H_\Lambda) \cong \bigotimes_{x \in \Lambda} \mathcal{B}(H_x) \qquad \text{(the finite-volume observable algebra)}$$

Notation: Denote by $\mathcal{P}_0(\Gamma) = \{ X \subset \Gamma : X \text{ is finite} \}$.

~~Although there is no well-defined "infinite tensor product of Hilbert spaces"~~, there is a well-defined observable algebra associated to Γ. We introduce this as follows.

The algebra of quasi-local observables

Recall that:

If $\Lambda_0 \subset \Lambda \subset \Gamma$ are finite sets, then each $A \in \mathcal{A}_{\Lambda_0}$ can be identified with

$$A \otimes 1\!\!1_{\Lambda \setminus \Lambda_0} \in \mathcal{A}_\Lambda .$$

In this case, $\mathcal{A}_{\Lambda_0}$ can be identified with a $*$-subalgebra of $\mathcal{A}_\Lambda$. By an inductive limit, we set

$$\mathcal{A}_\Gamma^{loc} = \bigcup_{\Lambda \in \mathcal{P}_0(\Gamma)} \mathcal{A}_\Lambda$$

to be the algebra of <u>local</u> <u>observables</u>.

One checks that $\mathcal{A}_\Gamma^{loc}$ is a normed $*$-algebra but, in general, (specifically when $|\Gamma| = +\infty$) $\mathcal{A}_\Gamma^{loc}$ is not complete.

We set

$$\mathcal{A}_\Gamma = \overline{\mathcal{A}_\Gamma^{loc}}^{\|\cdot\|}$$

(Here we take the norm (operator norm) closure of $\mathcal{A}_\Gamma^{loc}$.)

to be the algebra of <u>quasi-local observables</u>.

$\mathcal{A}_\Gamma$ is a well-defined C^*-algebra. It is this C^*-algebra on which most of our models are well-defined in the thermodynamic limit.

Terminology

We will say that a sequence $\{\Lambda_n\}_{n\geq 1}$ with $\Lambda_n \in \mathcal{P}_0(\Gamma)$ for all $n \geq 1$ is an increasing, absorbing sequence of finite subsets of Γ if:

i) $\Lambda_1 \neq \phi$ (non-trivial)

ii) $\Lambda_n \subset \Lambda_{n+1}$ for all $n \geq 1$ (increasing)

iii) for each $x \in \Gamma$, $\exists n_x \geq 1$ s.t. $x \in \Lambda_{n_x}$ (absorbing)

Theorem (Existence of the Dynamics)
Let (Γ, d) be a regular metric space, F an F-function on (Γ, d), and $\Phi \in \mathcal{B}_F(\Gamma)$. Then for any increasing, absorbing sequence $\{\Lambda_n\}_{n\geq 1}$ of finite subsets of Γ, the limit

$$(*) \quad \tau_t^{\Phi}(A) = \lim_{n\to\infty} \tau_t^{\Lambda_n, \Phi}(A)$$

exists (in norm) for each $A \in \mathcal{A}_\Gamma^{loc}$ and $t \in \mathbb{R}$.
The convergence in $(*)$ is uniform for t in compact subsets of $\mathbb{R}$ and independent of the choice of increasing, absorbing sequence $\{\Lambda_n\}_{n\geq 1}$. The family $\{\tau_t^{\Phi}\}_{t\in\mathbb{R}}$ defines a strongly continuous, one parameter group of $*$-automorphisms of $\mathcal{A}_\Gamma$.

We call $\{\tau_t^\Phi\}_{t\in\mathbb{R}}$ the <u>infinite volume</u> <u>dynamics</u> associated to Φ.

The limit in $(*)$ above is referred to as the <u>thermodynamic</u> <u>limit</u> of the finite-volume dynamics.

discuss strong continuity ...

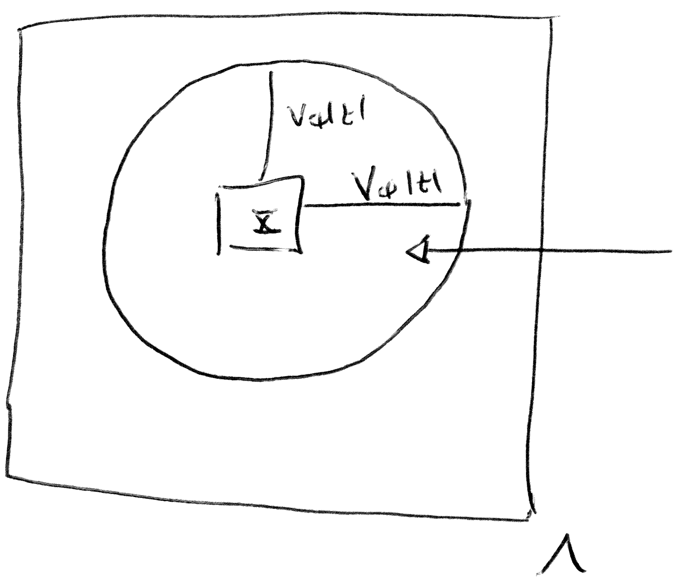
<u>Proof:</u>

The main idea of the proof is as follows.

Each $\Phi\in\mathcal{B}_F(\Gamma)$ satisfies a Lieb-Robinson bound.

Thus for any $A\in\mathcal{A}_\Gamma^{loc}$ with $\mathrm{supp}(A)=X\subset\Gamma$ finite, given $\Lambda\in\mathcal{P}_0(\Gamma)$ with $X\subset\Lambda$ one has that:

For each fixed $t\in\mathbb{R}$



$\mathrm{supp}(\tau_t^{\Lambda,\Phi}(A))$

$\subset B_X(v_\Phi|t|)$

$=\{x\in\Gamma : d(X,x)\le v_\Phi|t|\}$

In words, if v_Φ is the Lieb-Robinson velocity associated to $\Phi\in\mathcal{B}_F(\Gamma)$, then the support of $\tau_t^{\Lambda,\Phi}(A)$ is essentially contained in the ball of radius $v_\Phi|t|$ centered at X. Since v_Φ is independent of the finite volume $\Lambda\subset\Gamma$ on which the dynamics is defined, this suggests that the thermodynamic limit "should" exist for any $A\in\mathcal{A}_\Gamma^{loc}$.

Let us now prove this in detail.

Let $A \in \mathcal{A}_\Gamma^{loc}$ and let $X = \mathrm{supp}(A) \in \mathcal{P}_0(\Gamma)$.

Let $\{\Lambda_n\}_{n\geq 1}$ be an increasing, absorbing sequence of finite subsets of Γ and take $N \geq 1$ large enough so that $X \subset \Lambda_n$ for all $n \geq N$.

Our goal is to show that.
For each $t \in \mathbb{R}$ fixed, the sequence $\{\tau_t^{\Lambda_n,\Phi}(A)\}_{n\geq 1}$ is Cauchy in $\mathcal{A}_\Gamma^{loc}$. Given this, the limit

$$\lim_{n\to\infty} \tau_t^{\Lambda_n,\Phi}(A) = \tau_t^{\Phi}(A) \in \mathcal{A}_\Gamma = \overline{\mathcal{A}_\Gamma^{loc}}^{\|\cdot\|}$$

is well-defined. Note that $\mathrm{supp}(\tau_t^{\Lambda_n,\Phi}(A)) \subset X \cup \Lambda_n \in \mathcal{P}_0(\Gamma)$ and thus this sequence is contained in $\mathcal{A}_\Gamma^{loc}$.

Note further: our estimates will show that this sequence is uniformly Cauchy for t in compact subsets of $\mathbb{R}$.

For any $N \leq m \leq n$, one has that $(t \geq 0)$

$$\tau_t^{\Lambda_n,\Phi}(A) - \tau_t^{\Lambda_m,\Phi}(A) = \int_0^t \frac{d}{ds}\left(\tau_s^{\Lambda_n,\Phi}\left(\tau_{t-s}^{\Lambda_m,\Phi}(A)\right)\right) ds$$

$$= i \int_0^t \tau_s^{\Lambda_n,\Phi}\left(\left[H_{\Lambda_n} - H_{\Lambda_m}, \tau_{t-s}^{\Lambda_m}(A)\right]\right) ds$$

In this case, (again for $t \geq 0$) (6)

$$\| \tau_t^{\Lambda_n,\Phi}(A) - \tau_t^{\Lambda_m,\Phi}(A) \| \leq \sum_{\substack{Z \subset \Lambda_n: \\ Z \cap \Lambda_n \setminus \Lambda_m \neq \emptyset \\ Z \cap \Lambda_m \neq \emptyset}} \int_0^t \| [\tau_s^{\Lambda_m,\Phi}(A), \Phi(Z)] \| \, ds$$

<u>Note</u>:

i) any term $\Phi(Z)$ with $Z \subset \Lambda_m$ cancels in

$$H_{\Lambda_n} - H_{\Lambda_m}$$

ii) any term $\Phi(Z)$ with $Z \subset \Lambda_n \setminus \Lambda_m$ ~~cancels in~~ satisfies

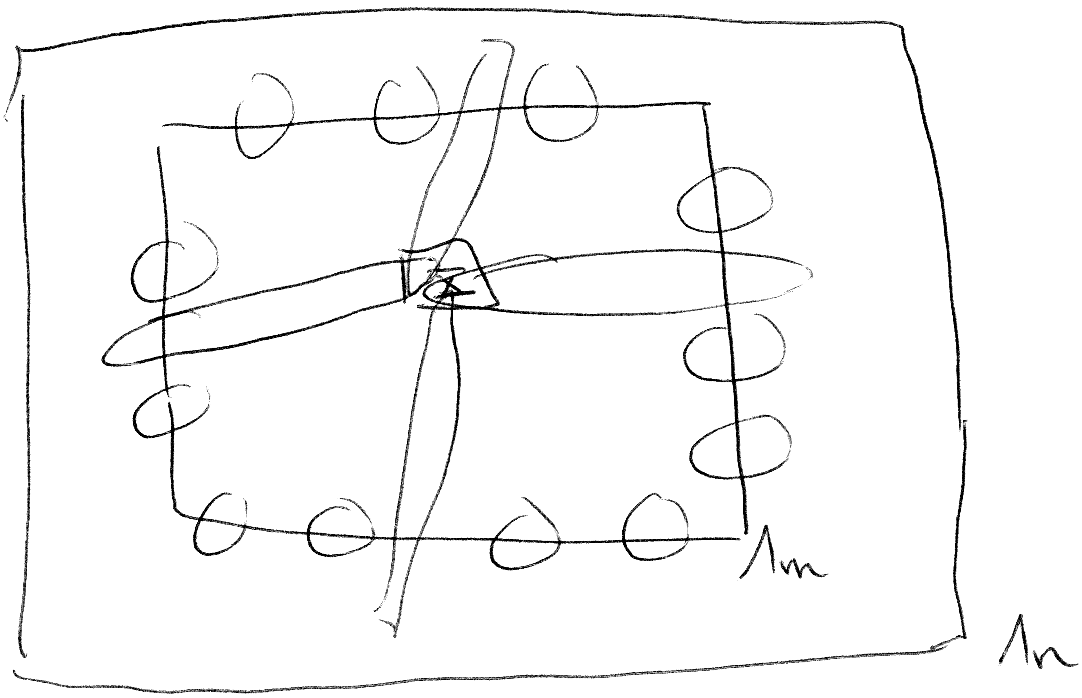
$$[\Phi(Z), \tau_{t-s}^{\Lambda_m}(A)] = 0.$$

In this case, only the terms indicated above "survive" the norm estimate.

Moreover, in the estimate above there are "two main" types of contributions

<u>Type 1</u>:
Boundary terms

<u>Type 2</u>:
Terms with large support



For type 2 terms, the Lieb-Robinson bounds are not useful. Otherwise, the relevant decay is governed by the LRB.

We now complete the bound:

$$\left\| \tau_t^{\Lambda_n,\Phi}(A) - \tau_t^{\Lambda_m,\Phi}(A) \right\| \le \sum_{\substack{Z \subset \Lambda_n: \\ Z \cap \Lambda_n \setminus \Lambda_m \neq \emptyset \\ Z \cap \Lambda_m \neq \emptyset}} \int_0^t \left\| \left[\tau_s^{\Lambda_m,\Phi}(A), \Phi(Z) \right] \right\| ds$$

$$= \sum\nolimits_1 + \sum\nolimits_2$$

where

$$\sum\nolimits_1 = \sum_{\substack{Z \subset \Lambda_n: \\ Z \cap \Lambda_n \setminus \Lambda_m \neq \emptyset \\ Z \cap \Lambda_m \neq \emptyset \\ Z \cap X = \emptyset}} \int_0^t \left\| \left[\tau_s^{\Lambda_m,\Phi}(A), \Phi(Z) \right] \right\| ds$$

"boundary terms"
i.e. LRB applies

$$\sum\nolimits_2 = \sum_{\substack{Z \subset \Lambda_n: \\ Z \cap \Lambda_n \setminus \Lambda_m \neq \emptyset \\ Z \cap X \neq \emptyset}} \int_0^t \left\| \left[\tau_s^{\Lambda_m,\Phi}(A), \Phi(Z) \right] \right\| ds$$

LRB does not apply but the terms $\Phi(Z)$ have large support.

we estimate these terms differently.

$$\Sigma_1 = \sum_{\substack{Z \subset \Lambda_n: \\ Z \cap \Lambda_n \setminus \Lambda_m \neq \phi \\ Z \cap \Lambda_m \neq \phi \\ Z \cap X = \phi}} \int_0^t \| [\tau_s^{\Lambda_m, \Phi}(A), \Phi(Z)] \| \, ds$$

use LRB's

$$\leq \sum_{\substack{Z \subset \Lambda_n: \\ Z \cap \Lambda_n / \Lambda_m \neq \phi \\ Z \cap \Lambda_m \neq \phi \\ Z \cap X = \phi}} \int_0^t \frac{2\|A\| \cdot \|\Phi(Z)\|}{C_F} \left(e^{2C_F \|\Phi\|_F s} - 1 \right) ds \sum_{x \in X} \sum_{z \in Z} F(d(x,z))$$

$$\leq \frac{2\|A\|}{C_F} \int_0^t \left(e^{2C_F \|\Phi\|_F s} - 1 \right) ds \sum_{x \in X} \sum_{z' \in \Lambda_n / \Lambda_m} \sum_{z \in \Lambda_n} F(d(x,z)) \sum_{\substack{Z \subset \Lambda_n: \\ z, z' \in Z}} \|\Phi(Z)\|$$

$$\leq \frac{2\|A\| \, \|\Phi\|_F}{C_F} \int_0^t \left(e^{2C_F \|\Phi\|_F s} - 1 \right) ds \sum_{x \in X} \sum_{z' \in \Lambda_n / \Lambda_m} \sum_{z \in \Lambda_n} F(d(x,z)) \, F(d(z,z'))$$

$$\leq 2 \cdot \|A\| \, \|\Phi\|_F \int_0^t \left(e^{2C_F \|\Phi\|_F s} - 1 \right) ds \sum_{x \in X} \sum_{z' \in \Lambda_n / \Lambda_m} F(d(x,z'))$$

For the other terms, we argue as follows:

$$\Sigma_2 = \sum_{\substack{Z \subset \Lambda_n: \\ Z \cap \Lambda_n \setminus \Lambda_m \neq \phi \\ Z \cap X \neq \phi}} \int_0^t \| [\tau_s^{\Lambda_m, \Phi}(A), \Phi(Z)] \| \, ds$$

$$\leq 2\|A\| \cdot t \sum_{x \in X} \sum_{z \in \Lambda_n \setminus \Lambda_m} \sum_{\substack{Z \subset \Lambda_n: \\ x, z \in Z}} \|\Phi(Z)\|$$

$$\leq 2\|\Phi\|_F \|A\| t \sum_{x \in X} \sum_{z \in \Lambda_n \setminus \Lambda_m} F(d(x,z))$$

Both of these terms have now been estimated similarly.

Note that: For each $x \in X$,

$$\sum_{z \in \Lambda_n \setminus \Lambda_m} F(d(x,z)) \to 0 \quad \text{as } n, m \to \infty$$

as F is uniformly summable on (Γ, d).

Thus $\displaystyle\sum_{x \in X} \sum_{z \in \Lambda_n \setminus \Lambda_m} F(d(x,z)) \to 0$ as $n, m \to \infty$

since $|X| < \infty$. Since the t-dependence in these estimates is explicit, it is clear that this convergence is also uniform for t in compact sets.

we have proven (*) with estimates that are uniform for t in compact subsets of $\mathbb{R}$.

To see that the limit is independent of the sequence, suppose that for some $A \in \mathcal{A}_\Gamma^{loc}$ and $t \in \mathbb{R}$, there were two limiting values:

$$\tau_t^{\Phi,1}(A) = \lim_{n\to\infty} \tau_t^{\Lambda_n^1,\Phi}(A)$$

$$\tau_t^{\Phi,2}(A) = \lim_{n\to\infty} \tau_t^{\Lambda_n^2,\Phi}(A)$$

Then $\forall \varepsilon > 0$, $\exists N \geq 1$ s.t.

$$\| \tau_t^{\Phi,1}(A) - \tau_t^{\Lambda_n^1,\Phi}(A)\| \leq \varepsilon/3$$

and

$$\| \tau_t^{\Phi,2}(A) - \tau_t^{\Lambda_n^2,\Phi}(A)\| \leq \varepsilon/3$$

for all $n \geq N$.

Take $n_2 \geq N$ and $n_1 \geq N$ s.t. $\Lambda_{n_2}^2 \subset \Lambda_{n_1}^1$ which can be done as $\{\Lambda_n\}_{n\geq 1}$ is increasing and absorbing.

Then $$\|\tau_t^{\Phi,1}(A) - \tau_t^{\Phi,2}(A)\| \leq \frac{\varepsilon}{3} + \|\tau_t^{\Lambda_{n_1}^1,\Phi}(A) - \tau_t^{\Lambda_{n_2}^2,\Phi}(A)\| + \frac{\varepsilon}{3}$$

By our previous argument, the middle term goes to zero as $n_1, n_2 \to \infty$. In this case, the two limits must coincide.

In the homework, we discuss how to regard (for each fixed $t \in \mathbb{R}$) the sequence $\{\tau_t^{\Lambda_n, \alpha}\}$ as a sequence of automorphisms of $\mathfrak{A}_\Gamma$. The proof of $(*)$ shows that this sequence of automorphisms converges strongly on a dense set. By a homework problem this sequence then converges on all of $\mathfrak{A}_\Gamma$ and its limit is an automorphism. Using the Bounded linear transformation theorem, the group property of the finite volume dynamics extends to the infinite volume dynamics. The proof of strong continuity is as follows.

Let $A \in \mathfrak{A}_\Gamma$. We wish to show that:

$$\lim_{t \to t_0} \|\tau_t(A) - \tau_{t_0}(A)\| = 0.$$

Fix $\varepsilon > 0$. Since $\mathfrak{A}_\Gamma = \overline{\mathfrak{A}_\Gamma^{loc}}^{\|\cdot\|}$, $\exists\, A_0 \in \mathfrak{A}_\Gamma^{loc}$ with

$\|A - A_0\| \leq \varepsilon/3$.

In this case,

$$\begin{aligned}
\|\tau_t(A) - \tau_{t_0}(A)\| &= \|\tau_t(A) - \tau_t(A_0) + \tau_t(A_0) - \tau_{t_0}(A_0) + \tau_{t_0}(A_0) - \tau_{t_0}(A)\| \\
&\leq \|\tau_t(A) - \tau_t(A_0)\| + \|\tau_t(A_0) - \tau_{t_0}(A_0)\| \\
&\qquad + \|\tau_{t_0}(A_0) - \tau_{t_0}(A)\| \\
&\leq 2 \cdot \|A - A_0\| + \|\tau_t(A_0) - \tau_{t_0}(A_0)\| \\
&\leq \frac{2}{3}\varepsilon + \|\tau_t(A_0) - \tau_{t_0}(A_0)\|
\end{aligned}$$

By compact uniform convergence, $\exists N \geq 1$ s.t.

$$\| \tau_t(A_0) - \tau_t^{\Lambda_n,\Phi}(A_0) \| \leq \frac{\varepsilon}{9} \quad \text{for all } t \in [t_0 - 1, t_0 + 1].$$

Thus

$$\| \tau_t(A_0) - \tau_{t_0}(A_0) \| \leq \| \tau_t(A_0) - \tau_t^{\Lambda_n,\Phi}(A_0) + \tau_t^{\Lambda_n,\Phi}(A_0) - \tau_{t_0}^{\Lambda_n,\Phi}(A_0) + \tau_{t_0}^{\Lambda_n,\Phi}(A_0) - \tau_{t_0}(A_0) \|$$

$$\leq \frac{2 \cdot \varepsilon}{9} + \| \tau_t^{\Lambda_n,\Phi}(A_0) - \tau_{t_0}^{\Lambda_n,\Phi}(A_0) \|$$

and finally,

$$\tau_t^{\Lambda_n,\Phi}(A_0) - \tau_{t_0}^{\Lambda_n,\Phi}(A_0) = \int_{t_0}^{t} \frac{d}{ds} \tau_s^{\Lambda_n,\Phi}(A_0)\, ds$$

$$= \int_{t_0}^{t} i\, \tau_s^{\Lambda_n,\Phi}([H_{\Lambda_n}, A_0])\, ds$$

$$\Rightarrow \| \tau_t^{\Lambda_n,\Phi}(A_0) - \tau_{t_0}^{\Lambda_n,\Phi}(A_0) \| \leq \sum_{\substack{Z \subset \Lambda_n: \\ Z \cap \mathrm{supp}(A_0) \neq \emptyset}} \| [\Phi(Z), A_0] \| \cdot |t - t_0|$$

$$\leq 2 \cdot \|A_0\| \cdot |t - t_0| \sum_{\substack{w \in \Lambda_n \\ x \in \mathrm{supp}(A_0)}} \sum_{\substack{Z \subset \Lambda_n: \\ w, x \in Z}} \|\Phi(Z)\|$$

$$\leq 2 \cdot \|\Phi\|_F \|A_0\| \cdot |t - t_0| \sum_{x \in \mathrm{supp}(A_0)} \sum_{w \in \Lambda_n} F(d(x,w))$$

$$\leq 2 \cdot \|F\| \cdot \|\Phi\|_F \|A_0\| |A_0| \cdot |t - t_0|$$

for $|t - t_0|$ sufficiently small, this estimate is less than $\varepsilon/9$!