Lecture 6

Last class, we introduced the tensor product of Hilbert spaces.
Today we talk about the tensor product of observables.

First, a useful theorem.

Theorem Let H_1 and H_2 be separable complex Hilbert spaces.
Let $\phi_1 \in H_1$ and $\phi_2 \in H_2$.

a) $\phi_1 \otimes \phi_2 = 0 \iff \phi_1 = 0$ or $\phi_2 = 0$.

b) Suppose $\phi_1 \neq 0$ and $\phi_2 \neq 0$. Then

$$\phi_1 \otimes \phi_2 = \psi_1 \otimes \psi_2 \iff \exists \lambda \in \mathbb{C} \setminus \{0\} \text{ for which } \psi_1 = \lambda \phi_1 \text{ and } \psi_2 = \lambda^{-1} \phi_2 .$$

Proof:

a) $$\phi_1 \otimes \phi_2 = 0 \iff \| \phi_1 \otimes \phi_2 \|_{H_1 \otimes H_2} = 0$$

$$\iff \|\phi_1\|_{H_1} \cdot \|\phi_2\|_{H_2} = 0$$

$$\iff \phi_1 = 0 \text{ or } \phi_2 = 0.$$

b) By a), if $\phi_1 \neq 0$ and $\phi_2 \neq 0$, then $\phi_1 \otimes \phi_2 \neq 0$.
In this case, if $\phi_1 \otimes \phi_2 = \psi_1 \otimes \psi_2$, then $\psi_1 \otimes \psi_2 \neq 0$
$\iff \psi_1 \neq 0$ and $\psi_2 \neq 0$.

Now, let $\{e_j\}_{j\geq 1}$ be an ONB of $\mathcal{H}_1$

and let $\{f_k\}_{k\geq 1}$ be an ONB of $\mathcal{H}_2$

write

$$\phi_1 = \sum_{j\geq 1} c_j e_j \quad \text{and} \quad \psi_1 = \sum_{j\geq 1} \tilde{c}_j e_j$$

and

$$\phi_2 = \sum_{k\geq 1} d_k f_k \quad \text{and} \quad \psi_2 = \sum_{k\geq 1} \tilde{d}_k f_k$$

Then using bilinearity

$$\phi_1 \otimes \phi_2 = \psi_1 \otimes \psi_2 \implies \sum_{j,k\geq 1} c_j d_k \, e_j \otimes f_k = \sum_{j,k\geq 1} \tilde{c}_j \tilde{d}_k \, e_j \otimes f_k$$

Using uniqueness of basis coefficients, we conclude

$$(*) \qquad c_j d_k = \tilde{c}_j \tilde{d}_k \quad \text{for all } j\geq 1 \text{ and all } k\geq 1.$$

Since $\psi_2 \neq 0$, $\exists\, k_0 \geq 1$ s.t. $\tilde{d}_{k_0} \neq 0$.

From this and $(*)$ we conclude

$$(**) \qquad c_j d_{k_0} = \tilde{c}_j \tilde{d}_{k_0} \quad \text{for all } j\geq 1$$

$$\implies \quad \tilde{c}_j = \left(\frac{d_{k_0}}{\tilde{d}_{k_0}}\right) c_j \quad \text{for all } j\geq 1.$$

<u>Note</u>: Since $\psi_1 \neq 0$, $d_{k_0} \neq 0$.

Taking $\lambda = \frac{d_{k_0}}{\tilde{d}_{k_0}} \in \mathbb{C} \setminus \{0\}$.

Clearly then

$$\psi_1 = \sum_{j \geq 1} \tilde{c}_j e_j = \sum_{j \geq 1} \lambda c_j e_j = \lambda \cdot \phi_1 \quad \text{as claimed.}$$

Note further:

if $\psi_1 \neq 0$, then $\exists j_0 \geq 1$ s.t. $\tilde{c}_{j_0} \neq 0$.

Using $(*)$ again, we find that

$$(***) \qquad c_{j_0} d_k = \tilde{c}_{j_0} \tilde{d}_k \quad \text{for all } k \geq 1.$$

$$\Rightarrow \qquad \tilde{d}_k = \left(\frac{c_{j_0}}{\tilde{c}_{j_0}} \right) d_k \quad \text{for all } k \geq 1$$

<u>Note</u>: Since $\psi_2 \neq 0$, $c_{j_0} \neq 0$. Moreover,

$$\psi_2 = \sum_{k \geq 1} \tilde{d}_k f_k = \sum_{k \geq 1} \left(\frac{c_{j_0}}{\tilde{c}_{j_0}} \right) d_k f_k = \left(\frac{c_{j_0}}{\tilde{c}_{j_0}} \right) \phi_2 .$$

Finally, evaluation of $(*)$ at $j = j_0$ and $k = k_0$ yields

$$c_{j_0} d_{k_0} = \tilde{c}_{j_0} \tilde{d}_{k_0} \Rightarrow \frac{c_{j_0}}{\tilde{c}_{j_0}} = \frac{\tilde{d}_{k_0}}{d_{k_0}} = \frac{1}{\lambda}$$

as claimed.

The next theorem is important in that it demonstrates well-definedness of a large class of observables on $B(H_1 \otimes H_2)$.

<u>Theorem</u> Let H_1 and H_2 be separable complex Hilbert spaces. For any $A \in B(H_1)$ and $B \in B(H_2)$, there exists a unique linear operator $A \otimes B \in B(H_1 \otimes H_2)$ for which

$$\square \qquad (A \otimes B)(\phi \otimes \psi) = (A\phi) \otimes (B\psi) \quad \text{for all } \phi \in H_1 \text{ and } \psi \in H_2.$$

Moreover, $\|A \otimes B\| = \|A\| \cdot \|B\|$.

- It is reasonable to expect that $\square$ defines a linear operator on $H_1 \otimes H_2$. This is part of what we will prove.
- <u>Recall</u>: For any $A \in B(H)$,

$$\|A\| = \sup_{\psi \in H / \{0\}} \frac{\|A\psi\|}{\|\psi\|}.$$

The final comment above is:

$$\|A \otimes B\|_{B(H_1 \otimes H_2)} = \|A\|_{B(H_1)} \cdot \|B\|_{B(H_2)}.$$

To prove this theorem, we need to review two facts from linear algebra.

Recall

Theorem (Gram-Schmidt)

Let V be a complex inner-product space. If $\{v_1, v_2, \ldots, v_n\} \subset V$ is a list of linearly independent vectors, then there exists an orthonormal set of vectors $\{e_1, e_2, \ldots, e_n\} \subset V$ for which

$$\text{span}(v_1, v_2, \ldots, v_k) = \text{span}(e_1, e_2, \ldots, e_k) \quad \text{for all } 1 \leq k \leq n.$$

Note: The requirement that the initial list of vectors is linearly independent is not necessary. If $\{v_1, v_2, \ldots, v_n\} \subset V$ is linearly dependent and nontrivial (i.e. not all zero), then there is some $1 \leq m < n$ and a subset $\{v_{i_1}, v_{i_2}, \ldots, v_{i_m}\} \subset \{v_1, v_2, \ldots, v_n\}$ which is linearly independent and satisfies

$$\text{span}(v_{i_1}, v_{i_2}, \ldots, v_{i_m}) = \text{span}(v_1, v_2, \ldots, v_n).$$

Gram-Schmidt applies to this smaller subset and usually suffices for applications.

- **Theorem** (Bounded linear transformations)

Let $\mathbb{X}$ be a normed space and $\mathbb{Y}$ be a complete normed space (ie a Banach space). Let $M \subset \mathbb{X}$ be a dense subspace. If $T : M \to \mathbb{Y}$ is a bounded linear map, then there is a unique bounded linear map $\overline{T} : \mathbb{X} \to \mathbb{Y}$ for which

- $\overline{T}(x) = T(x)$ for all $x \in M$

and

- $\|\overline{T}\| = \|T\|$.

In words, there is a unique bounded linear extension of any densely defined bounded linear operator. Importantly, the extension does not increase the operator norm.

Let us now describe, in words, how we will prove this theorem about the tensor product of operators.

We first show that the equation $\square$ allows us to define a unique linear operator on

$$M = \operatorname{span}\{\phi \otimes \psi : \phi \in H_1 \text{ and } \psi \in H_2\}.$$

It is clear that $M \subset \mathcal{H}_1 \otimes \mathcal{H}_2$ is a subspace. (spans are always subspaces...) Last class we showed that $\overline{M} = \mathcal{H}_1 \otimes \mathcal{H}_2$; i.e. M is a dense subspace.

In this case, we are in a position to apply the Bounded linear transformation theorem.

Proof:

Let $M = \text{span}(\phi \otimes \psi : \phi \in \mathcal{H}_1 \text{ and } \psi \in \mathcal{H}_2)$

This is clearly a dense subspace of $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Any vector $f \in M$ satisfies

$$f = \sum_{j=1}^{n} c_j \phi_j \otimes \psi_j \quad \text{with } \phi_j \in \mathcal{H}_1 \text{ and } \psi_j \in \mathcal{H}_2.$$

(Note: span means finite linear combinations)

In this case, any linear operator on M satisfying $\square$ has a clear action on f.

Let $C_{AB} : M \to \mathcal{H}_1 \otimes \mathcal{H}_2$. Take

$$C_{AB}(f) = C_{AB}\Big(\sum_{j=1}^{n} c_j \phi_j \otimes \psi_j\Big) = \sum_{j=1}^{n} c_j (A\phi_j) \otimes (B\psi_j).$$

This proof will demonstrate that

i) C_{AB} is well-defined on M

ii) C_{AB} is linear on M

iii) C_{AB} is bounded on M and $\|C_{AB}\| = \|A\| \cdot \|B\|$.

Once these 3 facts are verified, the bounded linear transformation theorem applies. We set

$A \otimes B = \overline{C_{AB}}$; and the theorem is proven.

We now prove i).

C_{AB} is well-defined on M.

The issue here is that many vectors in M can be represented as different finite sums. We need to show that C_{AB} takes any representation to the same value.

More precisely, suppose

$$\sum_{j=1}^{n_1} c_j\, \phi_j \otimes \psi_j = \sum_{i=1}^{n_2} d_i\, \phi_i' \otimes \psi_i'$$

we must show that

$$\sum_{j=1}^{n_1} c_j\, (A\phi_j) \otimes (B\psi_j) = \sum_{i=1}^{n_2} d_i\, (A\phi_i') \otimes (B\psi_i')$$

and therefore C_{AB} is well-defined.

Suppose

$$\sum_{j=1}^{n_1} c_j \phi_j \otimes \psi_j = \sum_{i=1}^{n_2} d_i \phi_i' \otimes \psi_i'$$

To each finite collection of vectors

$$\{\phi_j, \phi_i'\}_{i,j\geq 1} \subset H_1 \quad \text{and} \quad \{\psi_j, \psi_i'\}_{i,j\geq 1} \subset H_2$$

apply Gram-Schmidt.

In this case, there exists $\{\eta_k\}_{k=1}^{N_1} \subset H_1$ an ON set.

and $\{\theta_\ell\}_{\ell=1}^{N_2} \subset H_2$ an ON set

for which

$$\phi_j = \sum_{k=1}^{N_1} \alpha_k^j \eta_k \quad \text{and} \quad \phi_i' = \sum_{k=1}^{N_1} \beta_k^i \eta_k$$

and

$$\psi_j = \sum_{\ell=1}^{N_2} \gamma_\ell^j \theta_\ell \quad \text{and} \quad \psi_i' = \sum_{\ell=1}^{N_2} \delta_\ell^i \theta_\ell$$

Bilinearity implies that

$$\phi_j \otimes \psi_j = \sum_{k,\ell\geq 1} \alpha_k^j \gamma_\ell^j \, \eta_k \otimes \theta_\ell$$

and

$$\phi_i' \otimes \psi_i' = \sum_{k,\ell\geq 1} \beta_k^i \delta_\ell^i \, \eta_k \otimes \theta_\ell$$

In which case, we conclude that

$$\sum_{j\geq 1} c_j \phi_j \otimes \psi_j = \sum_{i \geq 1} d_i \phi_i \otimes \psi_i$$

$$\Rightarrow \sum_{j \geq 1} c_j \sum_{k,l \geq 1} \alpha_k^j \gamma_l^j \eta_k \otimes \theta_l = \sum_{i \geq 1} d_i \sum_{k,l \geq 1} \beta_k^i \delta_l^i \eta_k \otimes \theta_l$$

$$\Rightarrow \sum_{k,l \geq 1} \Big(\sum_{j \geq 1} c_j \alpha_k^j \gamma_l^j \Big) \eta_k \otimes \theta_l = \sum_{k,l \geq 1} \Big(\sum_{i \geq 1} d_i \beta_k^i \delta_l^i \Big) \eta_k \otimes \theta_l$$

$\Rightarrow$ For all $k \geq 1$ and $l \geq 1$

$$\sum_{j \geq 1} c_j \alpha_k^j \gamma_l^j = \sum_{i \geq 1} d_i \beta_k^i \delta_l^i$$

as coefficients in these expansions must be unique.

We conclude that

$$\sum_{j=1}^{n_1} c_j (A\phi_j) \otimes (B\psi_j) = \sum_{j \geq 1} c_j \sum_{k,l} \alpha_k^j \gamma_l^j (A\eta_k) \otimes (B\theta_l)$$

$$= \sum_{k,l} \Big(\sum_{j \geq 1} c_j \alpha_k^j \gamma_l^j \Big) (A\eta_k) \otimes (B\theta_l)$$

$$= \sum_{k,l} \Big(\sum_{i \geq 1} d_i \beta_k^i \delta_l^i \Big) (A\eta_k) \otimes (B\theta_l)$$

$$= \sum_{i \geq 1} d_i \sum_{k \geq 1} \beta_k^i \sum_{l \geq 1} \delta_l^i (A\eta_k) \otimes (B\theta_l)$$

$$= \sum d_i (A\phi_i) \otimes (B\psi_i)$$

using bilinearity

This proves that for any A and B, C_{AB} is a well defined operator on M.

The fact that C_{AB} is linear follows readily. (check!).

We now show that C_{AB} is bounded on M.

<u>Claim</u>: $C_{AB} = C_{A\mathbb{1}} \cdot C_{\mathbb{1}B}$

where

$$C_{A\mathbb{1}}(\phi \otimes \psi) = A\phi \otimes \psi$$

and

$$C_{\mathbb{1}B}(\phi \otimes \psi) = \phi \otimes B\psi$$

for all $\phi \in H_1$ and $\psi \in H_2$.

Since the previous arguments hold for any A and B, it is clear that both $C_{A\mathbb{1}}$ and $C_{\mathbb{1}B}$ are well-defined and linear on M. Their product, i.e. composition, satisfies

$$(C_{A\mathbb{1}} \cdot C_{\mathbb{1}B})(\phi \otimes \psi) = C_{A\mathbb{1}}(\phi \otimes B\psi) = A\phi \otimes B\psi$$

$$= C_{AB}(\phi \otimes \psi)$$

and thus they agree on M.

We will prove:

$$\|C_{A\mathbb{1}}\| \leq \|A\| \quad \text{and} \quad \|C_{\mathbb{1}B}\| \leq \|B\|.$$

As a consequence, we find that

$$\|C_{AB}\| \leq \|C_{A\mathbb{1}}\| \cdot \|C_{\mathbb{1}B}\| \leq \|A\| \cdot \|B\|.$$

Let's prove that $C_{A\mathbb{1}}$ is bounded on $\mathcal{M}$.

Let $f \in \mathcal{M}$. Using Gram Schmidt, write

$$f = \sum_{k,\ell} c_{k\ell}\, \eta_k \otimes \theta_\ell$$

where $\{\eta_k\}$, $\{\theta_\ell\}$, and $\{\eta_k \otimes \theta_\ell\}$ are all orthonormal collections of vectors.

Then clearly $\|f\|^2 = \sum_{k,\ell} |c_{k,\ell}|^2$ (by Pythagorus on $H_1 \otimes H_2$)

<u>Now</u>

$$\|C_{A\mathbb{1}} f\|^2 = \Big\| \sum_{k,\ell} c_{k\ell} (A\eta_k) \otimes \theta_\ell \Big\|^2$$

$$= \Big\| \sum_{\ell} \Big(\sum_k c_{k\ell} A\eta_k \Big) \otimes \theta_\ell \Big\|^2$$

Pythagorus on $H_1 \otimes H_2$ $\longrightarrow$

$$= \sum_\ell \Big\| \Big(\sum_k c_{k\ell} A\eta_k \Big) \otimes \theta_\ell \Big\|^2$$

norm on $H_1 \otimes H_2$ $\longrightarrow$

$$= \sum_\ell \Big\| \sum_k c_{k\ell} A\eta_k \Big\|^2$$

linearity of A $\longrightarrow$

$$\leq \sum_\ell \|A\|^2 \cdot \Big\| \sum_k c_{k\ell} \eta_k \Big\|^2$$

Pythagorus on H_1 $\longrightarrow$

$$= \sum_\ell \|A\|^2 \cdot \sum_k |c_{k\ell}|^2$$

$$= \|A\|^2 \cdot \|f\|^2$$

$\Rightarrow \|C_{A\mathbb{1}}\| \leq \|A\|$ as desired.

The proof that

$$\|C_{\mathbb{1}}B\| \leq \|B\|$$

is left for homework.

We have shown that $\|C_{AB}\| \leq \|A\| \cdot \|B\|$.

valid since

To prove equality, we show the reverse inequality.

Let $\varepsilon > 0$.

There is $\phi_\varepsilon \in H_1$ and $\psi_\varepsilon \in H_2$ for which

- $\|A\phi_\varepsilon\| \geq \|A\| - \varepsilon$
- $\|B\psi_\varepsilon\| \geq \|B\| - \varepsilon$
- $\|\phi_\varepsilon\| = \|\psi_\varepsilon\| = 1$

use that the operator norm is defined as a sup.

$$\Rightarrow \quad \phi_\varepsilon \otimes \psi_\varepsilon \in \mathcal{M} \text{ and } \|\phi_\varepsilon \otimes \psi_\varepsilon\| = 1$$

$$\begin{aligned}\Rightarrow \quad \|C_{AB}\| \geq \|C_{AB}(\phi_\varepsilon \otimes \psi_\varepsilon)\| &= \|A\phi_\varepsilon \otimes B\psi_\varepsilon\| \\ &= \|A\phi_\varepsilon\| \cdot \|B\psi_\varepsilon\| \\ &\geq (\|A\| - \varepsilon)(\|B\| - \varepsilon)\end{aligned}$$

Here restrict attention to the case that $\|A\| > 0$ and $\|B\| > 0$.

Taking limit as $\varepsilon \to 0$, we find that

$$\|C_{AB}\| \geq \|A\| \cdot \|B\|$$

This completes the proof of the theorem.