Lecture 7

Last class:

<u>Theorem</u> Let H_1 and H_2 be separable complex Hilbert spaces. For any $A \in B(H_1)$ and $B \in B(H_2)$, there is a unique linear operator $A \otimes B \in B(H_1 \otimes H_2)$ for which

$$\Box \qquad (A \otimes B)(\phi \otimes \psi) = (A\phi) \otimes (B\psi)$$

holds for all $\phi \in H_1$ and $\psi \in H_2$. Moreover,

$$\|A \otimes B\| = \|A\| \cdot \|B\|.$$

<u>Proof:</u>

We set

$$M = \text{span}\{\phi \otimes \psi : \phi \in H_1 \text{ and } \psi \in H_2\}.$$

We saw that $M \subset H_1 \otimes H_2$ is a dense subspace.

Using $\Box$, we defined a linear map $C_{AB} : M \to H_1 \otimes H_2$ and we checked that

i) C_{AB} is well-defined

ii) $C_{AB} \in B(M)$ with $\|C_{AB}\| \leq \|A\| \cdot \|B\|$.

We will now show

$$\|C_{AB}\| \geq \|A\| \cdot \|B\|.$$

So we have that

$$\|C_{A,B}\| = \|A\| \cdot \|B\|.$$

Using the BLT Theorem, there is a unique linear operator

$A \otimes B = \overline{C_{AB}} \in B(H_1 \otimes H_2)$ with $\|A \otimes B\| = \|A\| \cdot \|B\|$

as claimed.

To see this, we argue as follows.

<u>Note</u>: If $A = 0$ or $B = 0$, then $C_{A,B} = 0$ by ii).

In this case, we are done.

Otherwise $A \neq 0$ and $B \neq 0$, and $0 < \min[\|A\|, \|B\|]$.

For any $0 < \varepsilon < \min[\|A\|, \|B\|]$, $\exists\ \phi_\varepsilon \in H_1$ and $\psi_\varepsilon \in H_2$

for which

$\|A\phi_\varepsilon\| \geq \|A\| - \varepsilon$, $\|B\psi_\varepsilon\| \geq \|B\| - \varepsilon$, and $\|\phi_\varepsilon\| = \|\psi_\varepsilon\| = 1$.

(this uses that $\|A\| = \sup\limits_{\substack{\psi \in H: \\ \psi \neq 0}} \dfrac{\|A\psi\|}{\|\psi\|}$.)

$$\begin{aligned}
\Rightarrow \|C_{A,B}\| &\geq \|C_{A,B}(\phi_\varepsilon \otimes \psi_\varepsilon)\| \\
&= \|(A\phi_\varepsilon) \otimes (B\psi_\varepsilon)\| \\
&= \|A\phi_\varepsilon\| \cdot \|B\psi_\varepsilon\| \\
&\geq (\|A\| - \varepsilon)(\|B\| - \varepsilon)
\end{aligned}$$

Taking limit as $\varepsilon \downarrow 0$, we see that $\|C_{A,B}\| \geq \|A\| \cdot \|B\|$ and we are done.

More properties of tensor products of operators.

<u>Proposition 1</u>: Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be separable complex Hilbert spaces.

a) The map $(\cdot,\cdot) : B(\mathcal{H}_1) \times B(\mathcal{H}_2) \to B(\mathcal{H}_1 \otimes \mathcal{H}_2)$

given by $(A, B) \mapsto A \otimes B$ is bilinear.

b) For all $A_1, A_2 \in B(\mathcal{H}_1)$ and $B_1, B_2 \in B(\mathcal{H}_2)$,

$$(A_1 A_2) \otimes (B_1 B_2) = (A_1 \otimes B_1)(A_2 \otimes B_2).$$

c) For all $A \in B(\mathcal{H}_1)$ and $B \in B(\mathcal{H}_2)$,

$$(A \otimes B)^* = A^* \otimes B^*.$$

d) If $A_n \to A$ in $B(\mathcal{H}_1)$ and $B_n \to B$ in $B(\mathcal{H}_2)$,

then $A_n \otimes B_n \to A \otimes B$ in $B(\mathcal{H}_1 \otimes \mathcal{H}_2)$.

The proofs of part a) and part d) are homework.

<u>Note</u>: Given b), if A and B are invertible, then $A \otimes B$ is invertible. In fact,

$$(A \otimes B)(A^{-1} \otimes B^{-1}) = \mathbb{1}.$$

(Here need "boundedly invertible" ...)

More on this in the homework.

Proof:

b) Let $\phi \in \mathcal{H}_1$ and $\psi \in \mathcal{H}_2$.

Note that

$$[(A_1A_2) \otimes (B_1B_2)](\phi \otimes \psi) = (A_1A_2\phi) \otimes (B_1B_2\psi)$$

$$= [(A_1 \otimes B_1)](A_2\phi \otimes B_2\psi)$$

$$= [(A_1 \otimes B_1)(A_2 \otimes B_2)](\phi \otimes \psi)$$

The above clearly extends to finite linear combinations and hence to the dense set $M = \text{span}\{\phi \otimes \psi : \phi \in \mathcal{H}_1 \text{ and } \psi \in \mathcal{H}_2\}$.

Thus these two bounded linear operators agree on a dense subspace. By the BLT Theorem, there is a unique extension to $B(\mathcal{H}_1 \otimes \mathcal{H}_2)$, and hence these operators agree.

c) Let $\phi, \tilde{\phi} \in \mathcal{H}_1$ and $\psi, \tilde{\psi} \in \mathcal{H}_2$.

Note that

$$\langle (A \otimes B)^* (\phi \otimes \psi), \tilde{\phi} \otimes \tilde{\psi} \rangle = \langle \phi \otimes \psi, (A \otimes B)(\tilde{\phi} \otimes \tilde{\psi}) \rangle$$

$$= \langle \phi \otimes \psi, (A\tilde{\phi}) \otimes (B\tilde{\psi}) \rangle$$

$$= \langle \phi, A\tilde{\phi} \rangle \cdot \langle \psi, B\tilde{\psi} \rangle$$

$$= \langle A^*\phi, \tilde{\phi} \rangle \cdot \langle B^*\psi, \tilde{\psi} \rangle$$

$$= \langle A^*\phi \otimes B^*\psi, \tilde{\phi} \otimes \tilde{\psi} \rangle$$

$$= \langle (A^* \otimes B^*)(\phi \otimes \psi) \ \tilde{\phi} \otimes \tilde{\psi} \rangle$$

From this, we conclude that

- For each fixed $\Phi \in \mathcal{H}_1$ and $\Psi \in \mathcal{H}_2$, the above equality holds for all $\tilde{\Phi} \in \mathcal{H}_1$ and $\tilde{\Psi} \in \mathcal{H}_2$.
- Thus the equality holds for all $\tilde{\Phi} \otimes \tilde{\Psi} \in \mathcal{H}_1 \otimes \mathcal{H}_2$ and then for all elements of $M = \operatorname{span}\{\tilde{\Phi} \otimes \tilde{\Psi}\}$ a dense subspace.
- This implies that $(A \otimes B)^*(\Phi \otimes \Psi) = (A^* \otimes B^*)(\Phi \otimes \Psi)$ for all $\Phi \in \mathcal{H}_1$ and $\Psi \in \mathcal{H}_2$. (as vectors)
- Arguing as before, these two bounded linear operators must then agree by the BLT Theorem.

Let us now consider self-adjoint operators.

By the previous proposition, if A and B are self-adjoint, then $A \otimes B$ is also self-adjoint. In general, the converse is not true. Here is a fact.

<u>Proposition 2</u>: Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be separable complex Hilbert spaces. Let $A \in B(\mathcal{H}_1)$ and $B \in B(\mathcal{H}_2)$.

a) If $A \otimes B \neq 0$ is self-adjoint and A (or B) is self-adjoint, then B (or A) is also self-adjoint.

b) If $A \otimes B \neq 0$ is self-adjoint, then there is a number $c \in \mathbb{C}/\{0\}$ for which

$$\tilde{A} = cA \quad \text{and} \quad \tilde{B} = \frac{1}{c} B \quad \text{are self-adjoint operators.}$$

Proof:

a) Suppose $A \otimes B \neq 0$ is self-adjoint and B is self-adjoint.
Note: $B \neq 0$ (since otherwise $A \otimes B = 0$) and thus there is $\psi \in H_2/\{0\}$ for which $B\psi \neq 0$.

Note then that: For any $\phi \in H_1$

$$(A\phi) \otimes (B\psi) = (A \otimes B)(\phi \otimes \psi) = (A \otimes B)^*(\phi \otimes \psi)$$
$$= (A^* \otimes B^*)(\phi \otimes \psi)$$
$$= (A^*\phi) \otimes (B\psi)$$

$$\Rightarrow \quad (A - A^*)\phi \otimes B\psi = 0$$

$$\Rightarrow \quad A\phi = A^*\phi \quad \text{for all } \phi \in H_1$$

$\Rightarrow$ A is self-adjoint.

The other case is almost identical.

b) Suppose $A \otimes B \neq 0$ is self-adjoint.

If A (or B) is self-adjoint, then by the previous result this holds with the choice of $c = 1$.

Let us now assume that neither A nor B are self-adjoint.

Choose $\psi \in \mathcal{H}_2$ for which

- $B\psi \neq 0$ (true since $B \neq 0$)
- $\langle B\psi, \psi \rangle \neq 0$ (check!)

For any such choice, let $\phi_1, \phi_2 \in \mathcal{H}_1$.

Note that

$$\langle A\phi_1, \phi_2 \rangle \langle B\psi, \psi \rangle = \langle A\phi_1 \otimes B\psi, \phi_2 \otimes \psi \rangle$$

$$= \langle (A \otimes B)(\phi_1 \otimes \psi), \phi_2 \otimes \psi \rangle$$

$$= \langle \phi_1 \otimes \psi, (A \otimes B)(\phi_2 \otimes \psi) \rangle$$

$$= \langle \phi_1, A\phi_2 \rangle \langle \psi, B\psi \rangle$$

Then for $c = \langle \psi, B\psi \rangle \neq 0$, we conclude that

$$\langle cA\phi_1, \phi_2 \rangle = \langle \phi_1, cA\phi_2 \rangle \quad \text{for all } \phi_1, \phi_2 \in \mathcal{H}_1$$

$\Rightarrow$ $\tilde{A} = cA$ is self-adjoint.

Moreover, since

$$A \otimes B = \tilde{A} \otimes \tilde{B} \quad \text{with } \tilde{B} = \tfrac{1}{c} B$$

we conclude that $\tilde{B}$ is self-adjoint by part a).

The following is a useful observation.

Proposition 3

Let $A \in B(\mathbb{C}^m)$ and $B \in B(\mathbb{C}^n)$ both be self-adjoint.

Let $\{\lambda_j\}_{j=1}^m$ and $\{\mu_k\}_{k=1}^n$ be the eigenvalues of A and B respectively, each counted according to multiplicity.

Then

i) The self-adjoint operator $A \otimes B$ has eigenvalues $\{\lambda_j \mu_k\}_{j,k \geq 1}$

ii) The self-adjoint operator

$$A \otimes \mathbb{1} + \mathbb{1} \otimes B$$

has eigenvalues $\{\lambda_j + \mu_k\}_{j,k \geq 1}$.

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proof:

By the spectral theorem, there are orthonormal bases $\{e_j\}_{j=1}^m$ and $\{f_k\}_{k=1}^n$ of $\mathbb{C}^m$ and $\mathbb{C}^n$ respectively for which:

$$A = \sum_{j=1}^m \lambda_j P_{e_j} = \sum_{j=1}^m \lambda_j |e_j\rangle\langle e_j|$$

and

$$B = \sum_{k=1}^n \mu_k P_{f_k} = \sum_{k=1}^n \mu_k |f_k\rangle\langle f_k|$$

It is then clear that $\{e_j \otimes f_k\}_{j,k \geq 1}$ is an orthonormal basis for $\mathbb{C}^m \otimes \mathbb{C}^n$.

Moreover

$$(A \otimes B)(e_j \otimes f_k) = \lambda_j \mu_k (e_j \otimes f_k)$$

and

$$[A \otimes \mathbb{1} + \mathbb{1} \otimes B](e_j \otimes f_k) = (\lambda_j + \mu_k)(e_j \otimes f_k)$$

with each relation holding for all $1 \leq j \leq m$ and $1 \leq k \leq n$.

Since $\{e_j \otimes f_k\}_{j,k \geq 1}$ is an ONB, this is a complete description of the spectrum of these operators.

Note: We also know the corresponding eigenvectors!

Another useful observation about the tensor product of operators..

Theorem: Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be separable complex Hilbert spaces. Let $A \in B(\mathcal{H}_1)$ and $B \in B(\mathcal{H}_2)$. Denote by

$$H = A \otimes \mathbb{1} + \mathbb{1} \otimes B \in B(\mathcal{H}_1 \otimes \mathcal{H}_2).$$

One has that

$$e^H = e^A \otimes e^B.$$

The above theorem holds for arbitrary bounded operators A and B.

Recall: If H is a Hilbert space and $A \in B(H)$, then

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} .$$

- This series converges in $B(H)$.
- When A is self-adjoint, this definition coincides with the operator defined via functional calculus.
- This is how the equation

$$e^H = e^A \otimes e^B$$

is to be understood.

An immediate corollary of this theorem is:

Corollary Let H_1 and H_2 be separable complex Hilbert spaces.

Let $A \in B(H_1)$ with A self-adjoint.
Let $B \in B(H_2)$ with B self-adjoint.
For any $t \in \mathbb{R}$,

$$e^{-itH} = e^{-itA} \otimes e^{-itB}$$

where $H = A \otimes \mathbb{1} + \mathbb{1} \otimes B$.

In words, this corollary says that the Schrödinger evolution corresponding to H is just the tensor product of the Schrödinger evolutions of A and B.

This is because $H = A \otimes \mathbb{1} + \mathbb{1} \otimes B$ is a "non-interacting" Hamiltonian.

The proof of the corollary is simple.

Note that for any $t \in \mathbb{R}$

$$-itH = (-itA) \otimes \mathbb{1} + \mathbb{1} \otimes (-itB)$$

and both $(-itA)$ and $(-itB)$ are bounded if A and B are bounded and t is fixed.

To prove the theorem, first recall a simple fact.

For any $x, y \in \mathbb{R}$,

$$e^{x+y} = e^x \cdot e^y .$$

One can prove this with Taylor series:

$$e^{x+y} = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

binomial expansion

$$= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{1}{\cancel{n!}} \frac{\cancel{n!}}{k!\,(n-k)!} x^k y^{n-k}$$

$\boxed{m = n-k}$

$$= \sum_{k=0}^{\infty} \frac{x^k}{k!} \cdot \sum_{m=0}^{\infty} \frac{y^m}{m!} = e^x \cdot e^y$$

We use this fact as follows.

Let $X, Y \in B(H)$ and suppose $[X,Y]=0$ (i.e. $XY = YX$.)

Then
$$e^{X+Y} = e^{X} \cdot e^{Y}.$$

(Since the bounded operators commute, the analogue of the binomial expansion holds. In this case, the previous argument holds for operators too!)

We apply this as follows.

Let $X = A \otimes 1\!\!1$ and $Y = 1\!\!1 \otimes B$.

Then
$$X Y = (A \otimes 1\!\!1)(1\!\!1 \otimes B) = A \otimes B = (1\!\!1 \otimes B)(A \otimes 1\!\!1) = YX.$$

Thus
$$e^{H} = e^{A \otimes 1\!\!1} \cdot e^{1\!\!1 \otimes B}.$$

We will now show that

$$e^{A \otimes 1\!\!1} = e^{A} \otimes 1\!\!1 \quad \text{and similarly} \quad e^{1\!\!1 \otimes B} = 1\!\!1 \otimes e^{B}.$$

Given this

$$e^{H} = e^{A \otimes 1\!\!1} \cdot e^{1\!\!1 \otimes B} = (e^{A} \otimes 1\!\!1)(1\!\!1 \otimes e^{B}) = e^{A} \otimes e^{B}$$

and we are done!

We prove this result for $A \otimes \mathbb{1}$.

The result for $\mathbb{1} \otimes B$ is quite similar.

$$e^{A\otimes\mathbb{1}} = \sum_{n=0}^{\infty} \frac{(A\otimes\mathbb{1})^n}{n!}$$

$$= \lim_{N\to\infty} \sum_{n=0}^{N} \frac{(A\otimes\mathbb{1})^n}{n!} \qquad \leftarrow \text{just calculate.}$$

$$= \lim_{N\to\infty} \sum_{n=0}^{N} \frac{A^n\otimes\mathbb{1}}{n!} \qquad \leftarrow \text{bilinearity}$$

$$= \lim_{N\to\infty} \left(\sum_{n=0}^{N} \frac{A^n}{n!}\right)\otimes\mathbb{1}$$

$$= e^{A}\otimes\mathbb{1} \qquad \leftarrow \text{use Proposition 1 d).}$$

This completes the proof.