Second Quantization

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1 Fock Space

In the following paper, we go over a mathematical treatment of second quantization and, in the last section, we will discuss the way that it appears in many physics text books. The state of an n particle system is given by the Hilbert space $\mathcal{H} = \mathcal{L}^2(\mathbb{R}^{n\nu})$. To have variable number of particles, we introduce the Fock space defined as the direct sum

$$\mathcal{F}(\mathcal{H}) := \bigoplus_{n\geq 0} \mathcal{L}^2(\mathbb{R}^{n\nu}) = \mathbb{C} \oplus \mathcal{L}^2(\mathbb{R}^{\nu}) \oplus \mathcal{L}^2(\mathbb{R}^{2\nu}) \oplus \dots$$

Mathematically, this is the tensor algebra of $\mathcal{H}$, equipped with a Hilbert space inner product. We can rewrite this in terms of $\mathcal{H}$

$$\mathcal{H} = \mathbb{C} \oplus \mathcal{H}^{(1)} \oplus \mathcal{H}^{(2)} \oplus \mathcal{H}^{(3)} \dots$$

where $\mathcal{H}^{(n)} := \mathcal{H} \otimes \cdots \otimes \mathcal{H}$ (n times). It is not very hard to verify that $\mathcal{L}^2(\mathbb{R}^{2\nu}) = \mathcal{L}^2(\mathbb{R}^{\nu}) \otimes \mathcal{L}^2(\mathbb{R}^{\nu})$, and in general $\mathcal{L}^2(\mathbb{R}^{n\nu}) = \mathcal{L}^2(\mathbb{R}^{\nu})^{(n)}$. Elements of $\mathcal{F}$ are sequences of the form

$$\psi = \{\psi^{(n)}\}_{n\geq 0} = (\psi^{(0)}, \psi^{(1)}, \psi^{(2)}, \dots)$$

with $\psi^{(0)} \in \mathbb{C}$ and $\psi^{(n)} \in \mathcal{H}^{(n)}$. For $\phi, \psi \in \mathcal{F}$, the inner product is defined by

$$\langle \psi, \phi \rangle := \sum_{n\geq 0} \left\langle \psi^{(n)}, \phi^{(n)} \right\rangle$$

and the corresponding norm is given by $\|\psi\|^2 = \sum_{n\geq 0} \|\psi^{(n)}\|^2$. The state $(1, 0, 0, \dots) \in \mathcal{F}$ represents the vacuum, and it is denoted by Ω.

2 Bose and Fermi Statistics

For the Fock space corresponded to the Hilbert space $\mathcal{H} = \mathcal{L}^2(\mathbb{R}^{n\nu})$, there are two type of restrictions imposed by quantum statistics. These restrictions are arising from the effect of permuting the particle positions (or in tensor form permuting individual factors in a tensor product).

The first, arises when components ψ in each $\mathcal{H}^{(n)}$ is symmetric under interchange of coordinates, these are called **bosons**. The second case arises when components of ψ are anti-symmetric under interchange of coordinates (they will differ by a minus sign when transformed under an odd permutation), a particles with this property is called a **fermion**. The subspaces of $\mathcal{F}$, corresponded to bosons and fermions are denoted by $\mathcal{F}_+$ and $\mathcal{F}_-$ respectively.

The projections onto $\mathcal{F}_\pm$, for a simple tensor in $\mathcal{H}^{(n)}$ is given by

$$P_+(f_1 \otimes f_2 \otimes \cdots \otimes f_n) := \frac{1}{n!} \sum_\sigma f_{\sigma(1)} \otimes f_{\sigma(2)} \otimes \cdots \otimes f_{\sigma(n)}$$

$$P_-(f_1 \otimes f_2 \otimes \cdots \otimes f_n) := \frac{1}{n!} \sum_\sigma sgn(\sigma)\, f_{\sigma(1)} \otimes f_{\sigma(2)} \otimes \cdots \otimes f_{\sigma(n)}$$

where the sum is taken over all permutations. Also, $sgn(\sigma)$ is defined to be the determinant of the matrix constructed by applying σ to the columns of the identity matrix I_n.

Remark 1 *$P_\pm$ are bounded operators with norm equal to 1 as projections*

Now, the bose and fermi Fock subspaces can be defined precisely by $\mathcal{F}_+ := P_+(\mathcal{F})$ and $\mathcal{F}_- := P_-(\mathcal{F})$

3 Second Quantization of Operators

In the following section, let U be a unitary operator defined on $\mathcal{H}$. Also, assume that the self adjoint operator H is the infinitesimal generator of the unitary evolution, i.e

$$U_t = e^{itH}$$

To any unitary operator U on $\mathcal{H}$, one can correspond a unitary operator on $\mathcal{F}(\mathcal{H})$. In the case of two non-interacting particles, one may naturally define

$$\Gamma(U)(f \otimes g) := Uf \otimes Ug$$

This leads to the following definition

Definition 1 *For a unitary operator U on $\mathcal{H} = \mathcal{L}^2(\mathbb{R}^\nu)$, let $U_0 = 1$ and for the bosonic and fermi subspaces define U_n by*

$$U_n(P_\pm(f_1 \otimes \cdots \otimes f_n)) := P_\pm(Uf_1 \otimes \cdots \otimes Uf_n)$$

then $\Gamma(U)$, the second quantization of U on $\mathcal{F}(\mathcal{H})$, is defined by action of U_n on components $\mathcal{H}^{(n)}$, i.e

$$\Gamma(U) := \bigoplus_{n \geq 0} U_n$$

Now, we want to define second quantization of a self adjoin operator, as the generator of a unitary evolution (we assume that H is defined on a dense subset of $\mathcal{L}^2(\mathbb{R}^\nu)$). To have a natural definition, we need to examine the expression

$$\frac{d}{dt}\Big|_{t=0} U_t f \otimes U_t g$$

Recall, if B is a bilinear map, for vector valued functions f and g and small ϵ, we have

$$B\Big(f(\epsilon), g(\epsilon)\Big) \approx B\Big(f(0) + \epsilon f'(0), g(0) + \epsilon g'(0)\Big) = B\Big(f(0), g(0)\Big)$$

$$+\epsilon \left[B\Big(f'(0), g(0)) + B(f(0), g'(0)\Big)\right] + \epsilon^2 B\Big(f'(0), g'(0)\Big)$$

which suggests that

$$\frac{d}{dt}\Big|_{t=0} U_t f \otimes U_t g = iHf_1 \otimes f_2 + f_1 \otimes iHf_2$$

as a consequence, we have motivated the following definition:

Definition 2 *For a self adjoin operator H on $\mathcal{H} = \mathcal{L}^2(\mathbb{R}^\nu)$, let $H_0 = 0$ and define*

$$H_n(P_\pm(f_1 \otimes \cdots \otimes f_n)) := P_\pm(\sum_{i=1}^{n} f_1 \otimes \cdots \otimes Hf_i \otimes \cdots \otimes f_n)$$

then $d\Gamma(H)$, the second quantization of H on $\mathcal{F}(\mathcal{H})$, is defined

$$d\Gamma(U) := \overline{\bigoplus_{n\geq 0} H_n}$$

where the bar represents the self adjoin closure.

Now, if U_t is a strongly continous one-parameter group of evolution, then

$$\Gamma(U_t) = e^{itd\Gamma(H)}$$

4 The number operator

The number operator N, is defined as the second quntization of the identity operator on $\mathcal{H}$

$$N = d\Gamma(1)$$

$$N\psi = \{n\psi^{(n)}\}$$

with the domain

$$D(N) = \{\psi \in \mathcal{F} \ : \ \sum_{n\geq 0} n^2 \|\psi^{(n)}\| < \infty\}$$

To understand this operator, let see how it acts on a two particle state

$$d\Gamma(1)\, f \otimes g = 1f \otimes g + f \otimes 1g = 2f \otimes g$$

and in general, any n-particle state in $\mathcal{H}^{(n)}$ will be an eigenvector of N, with eigenvalue n.

5 The creation and Annihilation Operators

Definition 3 *The creation operator $a^\dagger(f)$ is defined by*

$$a^\dagger(f)\, f_1 \otimes \cdots \otimes f_n = \sqrt{n+1}\, f \otimes f_1 \otimes \cdots \otimes f_n$$

and the annihilation operator $a(f)$ by

$$a(f) f_1 \otimes \cdots \otimes f_n = \sqrt{n}\, \langle f, f_1 \rangle\, f_2 \otimes \cdots \otimes f_n$$

For bosons and fermions, creation and annihilation are defined by

$$a^\dagger_\pm(f) := P_\pm a^\dagger(f) P_\pm$$

$$a_\pm(f) := P_\pm a(f) P_\pm$$

The creation operator $a^\dagger(f)$ creates a particle with state f, and the annihilation operator annihilates a particle similarly. One can show that these operators are adjoint of each other. One particle states are given by

$$a^\dagger_\pm(f)\Omega$$

Using the fact that $P_-(f \otimes f) = 0$, we have $a^\dagger_-(f)\, a^\dagger_-(f) = 0$, this is the Pauli exclusion principle. On the Fock space the number operator N can be decomposed using operators N_f defined by

$$N_f := a^\dagger(f)a(f)$$

One can check that N_f counts the number of particles in state $f \in \mathcal{H}$, and for $\psi \in D(N)$

$$\langle \psi, N\psi\rangle = \sum_\alpha \langle \psi, N_{f_\alpha}\psi\rangle$$

where the sum is taken over an orthonormal basis of $\mathcal{H}$.

6 Position and Momentum Spaces

Recall our definition for creation operator $a^\dagger(f)$

$$a^\dagger(f)\, f_1 \otimes \cdots \otimes f_n = \sqrt{n+1}\, f \otimes f_1 \otimes \cdots \otimes f_n$$

$$a^\dagger(f)\,\Omega = f$$

In this section, we use above notation for a and $a^\dagger$ for the momentum space. Hence, $a^\dagger(\phi)$ creates a particle with momentum state $\phi(p)$. Using generalized functions, we can formally write a particle state with definite position x_0 and undetermined momentum:

$$\phi_{x_0}(p) = e^{-ip\cdot x_0}$$

to verify this, we can apply the inverse Fourier transform to get

$$\int_p e^{ip\cdot x}\phi_{x_0}(p)\, dp = \delta(x - x_0)$$

Using our notation

$$a^\dagger(e^{-ip\cdot x_0}) = e^{-ip\cdot x_0}$$

The operator $a^\dagger(e^{-ip\cdot x_0})$ is typically denoted simply by $a^\dagger_p$. Now, if we define a creation operator $\Psi^\dagger(x_0)$, using a new notation in position space, by

$$\Psi^\dagger(x_0)\Omega = \delta(x - x_0)$$

We observe that

$$\Psi^\dagger(x_0)\Omega = \int_p e^{ip\cdot x} a^\dagger(\phi_{x_0})\Omega\, dp$$

Hence, formally, one can write

$$\Psi^\dagger(x_0)\Omega = \left[\int\limits_p e^{ip\cdot x} a_p^\dagger \, dp\right]\Omega$$

so

$$\Psi^\dagger(x_0) = \int\limits_p e^{ip\cdot x} a_p^\dagger \, dp$$

We see that both $a_p^\dagger$ and $\Psi^\dagger(x_0)$ create a particle with definite position x_0, but in momentum space and position space respectively. In other words, $a^\dagger(\phi_{x_0})$ is the Fourier transform of $\Psi^\dagger(x_0)$. The same kind of relationship holds for annihilation operators

$$\Psi(x_0) = \int\limits_p e^{-ip\cdot x} a_p \, dp$$