MATH 541
HOMEWORK 1

FALL 2018

Throughout this assignment, for any $d \geq 1$ (an integer) let M_d be the collection of $d \times d$ matrices with complex entries. Said differently, let M_d denote the algebra of observables associated to a quantum system with an n-dimensional Hilbert space of states.

(1) Let V be a complex inner product space.

i) Show that V is a normed space when V is equipped with the *norm* induced by the inner product, i.e. show that the map $\|\cdot\| : V \to \mathbb{R}$ given by

$$\|x\| = \sqrt{\langle x, x\rangle} \quad \text{for all } x \in V$$

is a norm on V.

ii) Show that every finite-dimensional, complex inner product space is a Hilbert space when equipped with the norm from part i). In particular, show that, in this finite-dimensional case, the norm discussed above is complete.

(2) i) Show that the map $\langle\cdot,\cdot\rangle_{\mathrm{HS}} : M_d \times M_d \to \mathbb{C}$, with $(A, B) \mapsto \langle A, B\rangle_{\mathrm{HS}}$, given by

$$\langle A, B\rangle_{\mathrm{HS}} = \mathrm{Tr}[A^* B] \quad \text{for all } A, B \in M_d\,,$$

defines an inner-product on M_d. The map $\langle\cdot,\cdot\rangle_{\mathrm{HS}}$ is called the Hilbert-Schmidt inner product on M_d.

ii) Show that the norm induced by this inner-product, called the Hilbert-Schmidt norm, satisfies the following: For any orthonormal basis $\{u_j\}_{j=1}^d$ of $\mathbb{C}^d$, one has that

$$\|A\|_{\mathrm{HS}}^2 = \sum_{j=1}^{d} \|Au_j\|^2\,.$$

iii) Show that

$$\|A\| \leq \|A\|_{\mathrm{HS}}$$

where the quantity on the left-hand-side above is the operator norm of $A \in M_d$, i.e.

$$\|A\| = \sup_{\substack{\psi \in \mathbb{C}^d: \\ \psi \neq 0}} \frac{\|A\psi\|}{\|\psi\|}$$

(3) i) Show that for every state $\omega : M_d \to \mathbb{C}$, i.e., any normalized, positive linear functional on the observable algebra, there exists a $d \times d$ density matrix ρ, i.e. ρ is non-negative and has trace equal to one, for which

$$\omega(A) = \mathrm{Tr}[\rho A] \text{ for all } A \in M_d.$$

ii) Discuss uniqueness of the density matrix ρ found above.

iii) Show that ω is an extreme point of the set of states on M_d if and only if ω is a vector state.

(4) i) Show that $\rho \in M_2$ is a density matrix if and only if

$$\rho = \begin{pmatrix} r & \mu \\ \overline{\mu} & 1 - r \end{pmatrix}$$

for some $r \in [0, 1]$ and $\mu \in \mathbb{C}$ satisfying

$$|\mu|^2 \leq r(1 - r)\,.$$

ii) Show that $\rho \in M_2$ is a density matrix if and only if

$$\rho = \frac{1}{2}\mathbb{1} + \vec{x} \cdot \vec{\sigma}$$

with $\vec{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ satisfying $\|\vec{x}\| \leq 1$.

iii) Using the notation from part ii) above, show that the pure states on M_2 are in one-to-one correspondence with the unit vectors $\vec{x} \in \mathbb{R}^3$.

(5) We say that $A \in M_d$ is non-negative if

$$\langle x, Ax \rangle \geq 0 \quad \text{for all } x \in \mathbb{C}^d\,.$$

i) Show that $A \in M_d$ is non-negative if and only if there is some $B \in M_d$ for which $A = B^*B$. Conclude that any non-negative $A \in M_d$ is self-adjoint with non-negative eigenvalues.

We say that $A \in M_d$ is strictly positive if

$$\langle x, Ax \rangle > 0 \quad \text{for all } x \in \mathbb{C}^d \setminus \{0\}\,.$$

ii) Show that $A \in M_d$ is strictly positive if and only if $A \in M_d$ is non-negative and invertible.