MATH 541
HOMEWORK 2

FALL 2018

(1) Consider the collection of functions

$$\mathcal{H} = \{a : \mathbb{N}^2 \to \mathbb{C} \text{ with } \sum_{n,m\geq 1} |a(n,m)|^2 < \infty\}$$

This set is often written as $\mathcal{H} = \ell^2(\mathbb{N}^2)$. (The letter a is used for the functions since they are often thought of as sequences . . .) The goal of this exercise is to show that $\mathcal{H}$ is a Hilbert space with an explicit orthonormal basis.

i) Show that $\mathcal{H}$ is a complex vector space under the usual rules of arithmetic for functions. Moreover, show that

$$\langle a, b\rangle = \sum_{n,m\geq 1} \overline{a(n,m)} b(n,m) \quad \text{for all } a, b \in \mathcal{H}$$

is an inner product on $\mathcal{H}$.

ii) Show that $\mathcal{H}$, equipped as above, is a Hilbert space.

iii) Show that the collection of functions $\{e_{jk}\}_{j,k\geq 1}$ with

$$e_{jk}(n,m) = \delta_{jn} \cdot \delta_{km}$$

is an orthonormal basis of $\mathcal{H}$.

(2) Let $\mathcal{H}$ be a separable Hilbert space and $\{e_j\}_{j\geq 1} \subset \mathcal{H}$ be an orthonormal set. Prove that $\{e_j\}_{j\geq 1}$ is an orthonormal bases of $\mathcal{H}$ if and only if

$$\mathcal{H} = \overline{\text{span}(e_j : j \geq 1)}\,.$$

(3) Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be separable Hilbert spaces. Let $M \subset \mathcal{H}_1 \otimes \mathcal{H}_2$ be the following dense subspace:

$$M = \overline{\text{span}(\phi \otimes \psi : \phi \in \mathcal{H}_1 \text{ and } \psi \in \mathcal{H}_2)}\,.$$

We showed in class that the linear map $C_{\mathbb{1},B} : M \to \mathcal{H}_1 \otimes \mathcal{H}_2$ defined by requiring

$$C_{\mathbb{1},B}(\phi \otimes \psi) = \phi \otimes B\psi \quad \text{for all } \phi \in \mathcal{H}_1 \text{ and } \psi \in \mathcal{H}_2$$

is well-defined. Show that $\|C_{\mathbb{1},B}\| \leq \|B\|$ and carefully describe each line in the estimate.

(4) Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be separable Hilbert spaces.

i) Show that the map $(\cdot,\cdot) : \mathcal{B}(\mathcal{H}_1) \times \mathcal{B}(\mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ with $(A, B) \mapsto A \otimes B$ is bilinear.

ii) Show that if $A_n \to A$ in $\mathcal{B}(\mathcal{H}_1)$ and $B_n \to \mathcal{B}$ in $\mathcal{B}(\mathcal{H}_2)$, then $A_n \otimes B_n \to A \otimes B$ in $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$.

iii) Let $A \in \mathcal{B}(\mathcal{H}_1)$ and $B \in \mathcal{B}(\mathcal{H}_2)$. Show that $A \otimes B$ is boundedly invertible if and only if A and B are both boundedly invertible.

Recall: $A \in \mathcal{B}(\mathcal{H})$ is boundedly invertible if A^{-1} exists and $A^{-1} \in \mathcal{B}(\mathcal{H})$.

Hint: You may want to use that: If $A \in \mathcal{B}(\mathcal{H})$ is boundedly invertible then there is some positive number C for which

$$\|x\| \leq C\|Ax\| \quad \text{for all } x \in \mathcal{H}\,.$$

(5) Let $A \in \mathcal{B}(\mathbb{C}^m)$ and $B \in \mathcal{B}(\mathbb{C}^n)$. We proved in class that $\|A \otimes B\| = \|A\|\|B\|$ where the norm here is the operator norm. Check that the following is also true:

i) For A and B as above,

$$\mathrm{Tr}[A \otimes B] = \mathrm{Tr}[A]\mathrm{Tr}[B]\,.$$

ii) For A and B as above,

$$\|A \otimes B\|_1 = \|A\|_1\|B\|_1\,.$$

iii) For A and B as above,

$$\|A \otimes B\|_2 = \|A\|_2\|B\|_2\,.$$

Recall: For any matrix $A \in \mathcal{B}(\mathbb{C}^n)$, the trace norm of A, which we denote by $\|A\|_1$, is given by

$$\|A\|_1 = \sum_{j=1}^{n} \sigma_j(A)$$

where the numbers being added above are the singular values of A. Moreover, for any matrix $A \in \mathcal{B}(\mathbb{C}^n)$, the Hilbert-Schmidt norm of A, which we denote by $\|A\|_2$, is given by

$$\|A\|_2 = \sqrt{\mathrm{Tr}[A^*A]}\,.$$

Using the above as a definition, it is not hard to check that

$$\|A\|_2 = \sqrt{\sum_{j=1}^{n} \sigma_j(A)^2}.$$