MATH 541
HOMEWORK 3

FALL 2018

(1) Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be finite-dimensional (non-zero) complex Hilbert spaces. Let $A \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$.

i) Show that for any $C, D \in \mathcal{B}(\mathcal{H}_2)$, one has that

$$\mathrm{Tr}_{\mathcal{H}_1}[(\mathbb{1} \otimes C)A(\mathbb{1} \otimes D)] = C\,\mathrm{Tr}_{\mathcal{H}_1}[A]D\,.$$

ii) Show that the partial traces *preserve trace* in the sense that:

$$\mathrm{Tr}[\mathrm{Tr}_{\mathcal{H}_2}[A]] = \mathrm{Tr}[A] = \mathrm{Tr}[\mathrm{Tr}_{\mathcal{H}_1}[A]]\,.$$

(2) Let $m, n \geq 1$ be integers and set $\mathcal{H}_1 = \mathbb{C}^m$ and $\mathcal{H}_2 = \mathbb{C}^n$.

i) Let $A \in M_m$ and $B \in M_n$. Show that if A and B are both non-negative, then $A \otimes B \in M_{nm}$ is non-negative as well.

ii) Let $A \in M_m$ and $B \in M_n$. Show that if A and B are both strictly positive, then $A \otimes B \in M_{nm}$ is strictly positive as well.

iii) Let $A \in M_m$. Show that A is non-negative if and only if $\mathrm{Tr}[AB] \geq 0$ for all non-negative $B \in M_m$.

iv) Show that the partial traces preserve non-negativity: Let $A \in \mathcal{B}(\mathbb{C}^m \otimes \mathbb{C}^n) \equiv M_{mn}$. Show that if A is non-negative, then both $A_1 = \mathrm{Tr}_{\mathbb{C}^n}[A] \in M_m$ and $A_2 = \mathrm{Tr}_{\mathbb{C}^m}[A] \in M_n$ are non-negative as well.

(3) Let ρ be a density matrix on a finite dimensional Hilbert space $\mathcal{H}$.

i) Show that ρ is mixed if and only if $\mathrm{Tr}[\rho^2] < 1$.

ii) Suppose $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and let ρ be the density matrix corresponding to a pure state on $\mathcal{H}$. Show that both $\rho_1 = \mathrm{Tr}_{\mathcal{H}_2}[\rho]$ and $\rho_2 = \mathrm{Tr}_{\mathcal{H}_1}[\rho]$ are density matrices corresponding to pure states if and only if ρ corresponds to a product state. **Note:** Since ρ is assumed to be the density matrix corresponding to a pure state, $\rho = |\psi\rangle\langle\psi|$ and so ρ corresponds to a product state means that $\psi = \psi_1 \otimes \psi_2$.

iii) Let $\mathcal{H} = \mathbb{C}^n$ and take $\rho \in M_n$ to be any density matrix. Let $\{e_j\}_{j=1}^n$ be an orthonormal basis of $\mathbb{C}^n$ and take $\psi \in \mathbb{C}^n \otimes \mathbb{C}^n$ to be the vector given by

$$\psi = \sum_{j=1}^{n} e_j \otimes \rho^{1/2} e_j \, .$$

Denote by $\tilde{\rho} = |\psi\rangle\langle\psi|$ and check that $\tilde{\rho}$ is pure state on $\mathcal{B}(\mathbb{C}^n \otimes \mathbb{C}^n)$, i.e. check that ψ is a unit vector. Show that

$$\tilde{\rho}_2 = \mathrm{Tr}_{\mathbb{C}^n}[\tilde{\rho}] = \rho$$

and thus any density matrix can be found as the partial trace of the density matrix corresponding to a pure state. $\tilde{\rho}$ is sometimes referred to as the *purification* of ρ.

Hint: For any $B \in M_n$, calculate

$$\mathrm{Tr}[\tilde{\rho}(\mathbb{1} \otimes B)]$$

(4) Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be finite-dimensional (non-zero) complex Hilbert spaces and suppose $n = \dim(\mathcal{H}_1) = \dim(\mathcal{H}_2)$. Let ψ be a unit vector in $\mathcal{H}_1 \otimes \mathcal{H}_2$ and $\rho = |\psi\rangle\langle\psi|$ be the corresponding density matrix. Show that the following are equivalent:
i) Either ρ_1 or ρ_2 is maximally mixed,
and
ii) $\psi = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} e_j \otimes f_j$ where $\{e_j\}$ and $\{f_k\}$ are orthonormal bases of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively.

Unit vectors ψ which satisfy either i) or ii) above are called maximally entangled.

(5) Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be finite-dimensional (non-zero) complex Hilbert spaces. Let ρ be a density matrix on $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Denote by $S_E(\rho)$ the entanglement entropy of ρ.
i) Show that $S_E(\rho) = 0$ if and only if ρ corresponds to a separable state.
ii) Show that

$$0 \le S_E(\rho) \le \min[\ln(\dim(\mathcal{H}_1)), \ln(\dim(\mathcal{H}_2))]$$

Hint: First check that both of the above are true when ρ corresponds to a pure state. Then, show that the above remains true for when ρ is mixed.

(6) Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be finite-dimensional (non-zero) complex Hilbert spaces with orthonormal bases $\{e_j\}_{j=1}^m$ and $\{f_k\}_{k=1}^n$ respectively. Let $\psi \in \mathcal{H}_1 \otimes \mathcal{H}_2$ be a unit vector and denote by $A \in \mathbb{C}^{m\times n}$ the matrix of coefficients of ψ in the basis $\{e_j \otimes f_k\}_{j,k\geq 1}$, i.e. write

$$\psi = \sum_{j,k} \psi_{jk} e_j \otimes f_k \quad \text{and set} \quad A = \{\psi_{jk}\} \in \mathbb{C}^{m\times n}\,.$$

Denote by $\rho = |\psi\rangle\langle\psi|$ the density matrix associated to ψ.
i) Show that $\rho_1 = AA^*$.
ii) Show that $\rho_2 = A^*A$.
iii) Conclude that $S_E(\rho) = S(\rho_1) = S(\rho_2) = -\sum_n \sigma_n(A)^2 \ln(\sigma_n(A)^2)$.