MATH 541
HOMEWORK 4

FALL 2018

(1) Let $\mathcal{A}$ be a unital C^*-algebra. Let α be an automorphism of $\mathcal{A}$, i.e. α is a $*$-isomorphism from $\mathcal{A}$ to $\mathcal{A}$.
i) Show that $\alpha(\mathbb{1}) = \mathbb{1}$.
ii) Show that $\|\alpha(A)\| = \|A\|$ for all $A \in \mathcal{A}$. In class, this was stated as a corollary; prove it.

(2) Let α be a linear map $M_d \to M_d$. Show that α is an automorphism of M_d if and only if there exists a unitary $U \in M_d$, such that $\alpha(A) = U^*AU$, for all $A \in M_d$.

(3) Let $\mathcal{A}$ be a unital C^*-algebra. For each $n \geq 1$, let α_n be a linear map $\alpha_n : \mathcal{A} \to \mathcal{A}$. The sequence $\{\alpha_n\}_{n\geq 1}$ is said to converge strongly to a map $\alpha : \mathcal{A} \to \mathcal{A}$ if and only if

$$\lim_{n\to\infty} \alpha_n(A) = \alpha(A) \quad \text{for all } A \in \mathcal{A}\,.$$

i) Show that: if a sequence of automorphisms of $\mathcal{A}$, denoted by $\{\alpha_n\}$, converges strongly to a map α, then α is an automorphism of $\mathcal{A}$.
ii) Show that: if a sequence of automorphisms of $\mathcal{A}$, denoted by $\{\alpha_n\}$, satisfies

$$\lim_{n\to\infty} \alpha_n(A) = \alpha(A) \quad \text{for all } A \in \mathcal{A}_0$$

and $\mathcal{A}_0 \subset \mathcal{A}$ is a dense subset, then $\{\alpha_n\}_{n\geq 1}$ converges strongly to α. (Hence, α is an automorphism by i) above.)
iii) Let $\{\alpha_n\}$ be a sequence of automorphisms of $\mathcal{A}$. Show that: $\{\alpha_n\}$ converges strongly to the map α if and only if $\{\alpha_n^{-1}\}$ converges strongly to the map α^{-1}.

(4) Let $\mathcal{H}$ be a complex Hilbert space and $I \subset \mathbb{R}$ be an open interval.
i) Let $A, B : I \to \mathcal{B}(\mathcal{H})$ be continuous maps. Show that $C : I \times I \to \mathcal{B}(\mathcal{H})$ defined by

$$C(s,t) = A(s)B(t) \quad \text{for all } s,t \in I$$

is jointly continuous, i.e. let $(s_0, t_0) \in I \times I$ and show that

$$\lim_{(s,t)\to(s_0,t_0)} \|C(s,t) - C(s_0,t_0)\| = 0\,.$$

ii) Let $A, B : I \to \mathcal{B}(\mathcal{H})$ be differentiable maps. Show that $C : I \times I \to \mathcal{B}(\mathcal{H})$ defined by

$$C(s,t) = A(s)B(t) \quad \text{for all } s,t \in I$$

is separately differentiable, i.e. for each fixed $s \in I$, the map $t \mapsto C(s,t)$ is differentiable and for each fixed $t \in I$, the map $s \mapsto C(s,t)$ is differentiable.
iii) Let $A, B : I \to \mathcal{B}(\mathcal{H})$ be differentiable maps. Show that $C : I \to \mathcal{B}(\mathcal{H})$ defined by

$$C(t) = A(t)B(t) \quad \text{for all } t \in I$$

is differentiable and find a formula for its derivative.

(5) Let X be a Banach space and $I = [a,b] \subset \mathbb{R}$ be a compact interval. Consider the set of functions

$$\mathcal{F}([a,b],X) = \{f : [a,b] \to X \mid \|f\|_\infty = \sup_{a \le t \le b} \|f(t)\| < \infty\}\,.$$

It is easy to see that this set of functions is a vector space under the usual operations of arithmetic. Check that $\|\cdot\|_\infty$ is a norm on $\mathcal{F}([a,b],X)$ and that (with respect to this norm) $\mathcal{F}([a,b],X)$ is a Banach space.

(6) Let X be a Banach space and $I = [a,b] \subset \mathbb{R}$ be a compact interval. Denote by $\mathcal{S}([a,b],X) \subset \mathcal{F}([a,b],X)$ the set of step-functions on X, i.e. $f \in \mathcal{S}([a,b],X)$ if and only if there is an integer $n \ge 1$, a partition $\{t_j\}_{j=0}^n$ of $[a,b]$, and a collection of points $\{x_j\}_{j=0}^{n-1}$ with $x_j \in X$ for all $0 \le j \le n-1$ such that

$$f(t) = x_0 \chi_{[a,t_1]}(t) + \sum_{j=1}^{n-1} x_j \chi_{(t_j,t_{j-1}]}(t) \quad \text{for all } t \in [a,b]\,.$$

Show that the map $I : \mathcal{S}([a,b],X) \to X$ given by

$$I(f) = \sum_{j=0}^{n-1} (t_{j+1} - t_j) x_j$$

is a well-defined, linear map. This value $I(f) \in X$ is referred to as the Riemann integral of f in X. **Note:** Well-defined means that $I(f)$ is independent of the representation of f as a step function.

(7) Let X be a Banach space and $I = [a, b] \subset \mathbb{R}$ be a compact interval. Let $\overline{\mathcal{S}}([a, b], X)$ denote the norm closure of the subspace of step-functions $\mathcal{S}([a, b], X) \subset \mathcal{F}([a, b], X)$ defined in the previous problems. Show that the Riemann integral defined in class satisfies:
i) For any $a \leq \alpha < \beta \leq b$ and each $f \in \overline{\mathcal{S}}([a, b], X)$,

$$\left\| \int_\alpha^\beta f(t)\, dt \right\| \leq (\beta - \alpha) \sup\{\|f(t)\| : \alpha \leq t \leq \beta\} .$$

ii) For any $\alpha, \beta, \gamma \in [a, b]$ and each $f \in \overline{\mathcal{S}}([a, b], X)$,

$$\int_\alpha^\beta f(t)\, dt = \int_\alpha^\gamma f(t)\, dt + \int_\gamma^\beta f(t)\, dt .$$

iii) Show that for each $f \in \overline{\mathcal{S}}([a, b], X)$, the function $G : [a, b] \to X$ defined by setting

$$G(x) = \int_a^x f(t)\, dt \quad \text{for all } x \in [a, b]$$

is continuous, and moreover, $G(a) = 0$.
iv) Let Y be another Banach space and suppose $T \in \mathcal{B}(X, Y)$; the set of bounded linear maps from X to Y. Show that for each $f \in \overline{\mathcal{S}}([a, b], X)$ the image $Tf \in \overline{\mathcal{S}}([a, b], Y)$ and moreover,

$$T\left(\int_a^b f(t)\, dt \right) = \int_a^b (Tf)(t)\, dt .$$

v) Show that for each $f \in \overline{\mathcal{S}}([a, b], X)$ the map $t \mapsto \|f(t)\|$ is a real-valued element of $\overline{\mathcal{S}}([a, b], \mathbb{C})$ and moreover,

$$\left\| \int_a^b f(t)\, dt \right\| \leq \int_a^b \|f(t)\|\, dt .$$