MATH 541
HOMEWORK 5

FALL 2018

(1) Consider the metric space $(\mathbb{Z}, d)$ with $d(x,y) = |x-y|$ for all $x, y \in \mathbb{Z}$. Show that for any $a \geq 0$, the function $F_a : [0,\infty) \to (0,\infty)$ given by

$$F_a(r) = e^{-ar} \quad \text{for all } r \geq 0\,,$$

is not an F-function on $(\mathbb{Z}, d)$. **Note:** This result readily generalizes to any $(\mathbb{Z}^\nu, d)$ with $\nu \geq 1$, but focus in the one-dimensional case.

(2) i) Let $f : [0,\infty) \to [0,\infty)$ be a continuous function. Show that for each $n \geq 1$,

$$\int_0^x \int_0^{t_1} \cdots \int_0^{t_{n-1}} f(t_1)f(t_2)\cdots f(t_n) dt_n dt_{n-1} \cdots dt_1 = \frac{\left(\int_0^x f(t)\,dt\right)^n}{n!}$$

Hint: Prove this by induction with $n = 2$ being the base case.
ii) Let $\mathcal{H}$ be a Hilbert space and $A : [0,\infty) \to \mathcal{B}(\mathcal{H})$ be norm-continuous. Show that the function $W : [0,\infty) \to \mathcal{B}(\mathcal{H})$ given by

$$W(x) = \mathbb{1} + \sum_{n=1}^{\infty} \int_0^x \int_0^{t_1} \cdots \int_0^{t_{n-1}} A(t_1)A(t_2)\cdots A(t_n)\, dt_n dt_{n-1} \cdots dt_1$$

is well-defined for all $x \in [0,\infty)$ and provide an estimate on $\|W(x)\|$.

(3) In this problem, we investigate the Gronwall inequality.
i) Let $[a,b] \subset \mathbb{R}$. Suppose $f : [a,b] \to \mathbb{R}$ is continuous and

$$f(x) \leq \alpha + \int_a^x g(t)f(t)\,dt \quad \text{for all } x \in [a,b]\,,$$

where $\alpha \in \mathbb{R}$ and $g : [a,b] \to [0,\infty)$ is continuous. Show that

$$f(x) \leq \alpha e^{G(x)} \text{ for all } x \in [a,b] \text{ where } G(x) = \int_a^x g(t)\,dt\,.$$

This estimate is called a Gronwall inequality.
ii) Use this Gronwall inequality to show that the time-dependent Schrödinger equation has a unique solution. More precisely, let $I \subset \mathbb{R}$ be an open interval and $\mathcal{H}$ be a complex Hilbert space.
Let $H : I \to \mathcal{B}(\mathcal{H})$ satisfy

- $H(t)^* = H(t)$ for all $t \in I$, and
- $t \mapsto H(t)$ is norm continuous on I.

Then show that the $\mathcal{H}$-valued initial value problem:
Let $t_0 \in I$ and consider

$$\frac{d}{dt}\psi(t) = -iH(t)\psi(t) \quad \text{with } \psi(t_0) = \psi_0 \in \mathcal{H}$$

has a unique solution.
ii') A very similar argument show that the $\mathcal{B}(\mathcal{H})$-valued initial value problem:
Let $t_0 \in I$ and consider

$$\frac{d}{dt}U(t) = -iH(t)U(t) \quad \text{with } U(t_0) = \mathbb{1} \in \mathcal{B}(\mathcal{H})$$

also has a unique solution. Check this.
iii) Show that the $\mathcal{B}(\mathcal{H})$-valued initial value problem corresponding to the norm-preservation lemma has a unique solution. More precisely, let $I \subset \mathbb{R}$ be an open interval and $\mathcal{H}$ be a complex Hilbert space.
Let $A, B : I \to \mathcal{B}(\mathcal{H})$ satisfy

- $A(t)^* = A(t)$ for all $t \in I$, and
- both $t \mapsto A(t)$ and $t \mapsto B(t)$ are norm continuous on I.

Then show that the $\mathcal{B}(\mathcal{H})$-valued initial value problem:
Let $t_0 \in I$ and consider

$$\frac{d}{dt}f(t) = -i[A(t), f(t)] + B(t) \quad \text{with } f(t_0) = f_0 \in \mathcal{B}(\mathcal{H})$$

has a unique solution.