Some Abelian Varieties with Visible Shafarevich-Tate Groups

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Abstract

We give examples of abelian variety quotients of the modular jacobian $J_0(N)$ with nontrivial visible Shafarevich-Tate groups. In computing these examples we developed formulas for invariants of higher dimensional modular abelian varieties.

Introduction

In [4] Cremona and Mazur studied visibility of Shafarevich-Tate groups of elliptic curves $E \subset J_0(N)$. This paper extends some of these computations to higher dimensional $A \subset J_0(N)$. For each $N \leq 1001$ and $N = 1028$, 1061, we compute the analytic rank 0 new optimal A in $J_0(N)$ having nontrivial odd visible Shafarevich-Tate group, visible in the new part of $J_0(N)$. A total of 19 such A were found having Ш of order the squares of: $3, 5, 7, 3^2, 11, 13, 151$. Our computations say little about invisible elements of Ш, however see [1]. The algorithms developed for computing invariants of modular abelian varieties are also of some interest.

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Notation

Let $S_2(N) = S_2(\Gamma_0(N), \mathbf{C})$ be the space of cusp forms of weight 2 for the subgroup $\Gamma_0(N) \subset \mathrm{SL}_2(\mathbf{Z})$. Thus $S_2(N)$ can be identified with the differentials on the modular curve $X_0(N)$. Let $J_0(N)$ denote the jacobian of $X_0(N)$. Denote by $H_1(X_0(N), \mathbf{Z})$ the integral homology of $X_0(N)$. The Hecke algebra $\mathbf{T}$ acts in a compatible manner on $S_2(N)$, $H_1(X_0(N), \mathbf{Z})$ and $J_0(N)$. The letters f and g

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are reserved for newforms, i.e., normalized eigenforms for $\mathbf{T}$ in the complement of the old subspace of $S_2(N)$. Let $f_1, \ldots, f_d$ denote the Galois conjugates of f, and $I_f = \mathrm{Ann}(f)$ the annihilator of f in $\mathbf{T}$.

1 Optimal abelian varieties and newforms

An abelian variety quotient of $J_0(N)$ is called **optimal** if the kernel is connected. We associate to f an optimal quotient A_f of $J_0(N)$:

$$0 \to I_f J_0(N) \to J_0(N) \to A_f \to 0.$$

Let $H = H_1(X_0(N), \mathbf{Z})$, $S_2 = S_2(N)$, and recall that integration defines a non-degenerate pairing $S_2 \times H \to \mathbf{C}$, hence a map $H \to \mathrm{Hom}_{\mathbf{C}}(S_2, \mathbf{C})$. Composing with restriction to $S_2[I_f]$ defines a map $\Phi_f : H \to \mathrm{Hom}(S_2[I_f], \mathbf{C})$.

Theorem 1.1. *A_f is an abelian variety of dimension d with canonical L-series*

$$L(A_f, s) = \prod_{i=1}^{d} L(f_i, s).$$

The complex uniformization of the tori $A_f(\mathbf{C})$ and $A_f^\vee(\mathbf{C})$ is described by the following diagram

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
H[I_f] & \longrightarrow & H & \longrightarrow & \Phi_f(H) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Hom}(S_2, \mathbf{C})[I_f] & \longrightarrow & \mathrm{Hom}(S_2, \mathbf{C}) & \longrightarrow & \mathrm{Hom}(S_2[I_f], \mathbf{C}) \\
\downarrow & & \downarrow & & \downarrow \\
A^\vee(\mathbf{C}) & \longrightarrow & J_0(N)(\mathbf{C}) & \longrightarrow & A(\mathbf{C}) \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

in which the vertical columns are exact but the rows are not.

Proof. [16] and section 1.7 of [5]. □

1.1 The Birch and Swinnerton-Dyer Conjecture

Let $\mathcal{A}/\mathbf{Z}$ be the Neron model of $A = A_f$. The **Tamagawa number** c_p is the number of $\mathbf{F}_p$-rational components of the special fiber $\mathcal{A}_{\mathbf{F}_p}$. A basis $h_1, \ldots, h_d$ for the Neron differentials defines a measure μ on $A(\mathbf{R})$ and we let $\Omega_{\mathcal{A}} = \mu(A(\mathbf{R}))$. Let $w : H^1(\mathbf{Q}, A) \to \prod_v H^1(\mathbf{Q}_v, A)$ and set $Ш = \mathrm{Ker}(w)$. If $L(A, 1) \neq 0$ it is known [9] that $A(\mathbf{Q})$ and $Ш(A)$ are both finite. One then has the following fundamental and still open

Conjecture 1.2 (Birch, Swinnerton-Dyer, Tate).

$$L(A, 1) = \Omega_{\mathcal{A}} \cdot \frac{|Ш| \cdot \prod_{p \mid N} c_p}{|A(\mathbf{Q})| \cdot |A^\vee(\mathbf{Q})|}.$$

1.2 The Manin Constant

Let $S_f(\mathbf{C}) \subset S_2(N, \mathbf{C})$ denote the subspace of cusp forms spanned by the conjugates of f. There are two lattices in $S_f(\mathbf{C})$. One is the lattice $S_f(\mathbf{Z})$ of cusp forms with integer Fourier expansion at infinity. The other $S_f(\mathcal{A}/\mathbf{Z})$ is got by pulling back the Neron differentials defined above. The **Manin constant** is

$$c_f = [S_f(\mathbf{Z}) : S_f(\mathcal{A}/\mathbf{Z})].$$

We are aware of no examples of *newforms* f for which $c_f \neq 1$. It is reasonable to expect that one can extend methods known for elliptic curves (e.g., [11]) to show that c_f is at least coprime to $2N$.

Later in this paper we give a formula for $L(A_f, 1)/\Omega_f$ where Ω_f is computed using the lattice $S_f(\mathbf{Z})$. Thus our BSD special value is off by the Manin constant. For the remainder of this paper we officially assume

Conjecture 1.3. $c_f = 1$

1.3 Connecting Mordell-Weil and Shafarevich-Tate

Let $f, g \in S_2(N)$ be nonconjugate newforms and A_f, A_g the corresponding optimal quotients of $J_0(N)$. Let $\mathfrak{m} \subset \mathbf{T}$ be a maximal ideal such that $A_f^\vee[\mathfrak{m}] = A_g^\vee[\mathfrak{m}] \subset J_0(N)$. Let p be the residue characteristic of $\mathfrak{m}$. Assume that $p \nmid 2N \cdot \prod_{p|N} c_p c_p'$ where c_p' are the Tamagawa numbers of A_g.

Under hypothesis such as these we expect there to be a commutative diagram

$$\begin{array}{ccccccccc}
0 & \rightarrow & A_f(\mathbf{Q})/\mathfrak{m}A_f(\mathbf{Q}) & \rightarrow & X & \rightarrow & Ш(A_f)[\mathfrak{m}] & \rightarrow & 0 \\
 & & & & || & & & & \\
0 & \rightarrow & A_g(\mathbf{Q})/\mathfrak{m}A_g(\mathbf{Q}) & \rightarrow & X & \rightarrow & Ш(A_g)[\mathfrak{m}] & \rightarrow & 0
\end{array}$$

with exact rows. Here X is an abelian group, $H^1(\operatorname{Spec}\mathbf{Z}, A_f[\mathfrak{m}])$. A precise statement will not be given here as the purpose of this paper is merely to present a few algorithms and computational results.

2 Algorithms

2.1 Modular Symbols

Modular symbols give a presentation of the homology of the modular curve $X_0(N)$. Here we briefly review modular symbols for $\Gamma_0(N)$. More information on how to actually compute with them can be found in [3], [7], and [12].

Define the space of **modular symbols** $\underline{\mathcal{M}}(N, \mathbf{Z})$ to be the *free* abelian group generated by symbols $\{\alpha, \beta\}$ such that $\alpha, \beta \in \mathbf{P}^1(\mathbf{Q}) = \mathbf{Q} \cup \{\infty\}$ subject to the relations

$$\begin{aligned}
0 &= \{\alpha, \beta\} + \{\beta, \gamma\} + \{\gamma, \alpha\}, \\
\{\alpha, \beta\} &= \{g(\alpha), g(\beta)\}, \quad \text{all } g \in \Gamma_0(N).
\end{aligned}$$

The space of **boundary symbols** $\underline{\mathcal{B}}(N, \mathbf{Z})$ is the free abelian group generated by symbols $\{\alpha\}$, $\alpha \in \mathbf{P}^1(\mathbf{Q})$, modulo the relations

$$\{\alpha\} = \{g(\alpha)\}, \quad \text{all } g \in \Gamma_0(N).$$

The **cuspidal symbols** $\underline{\mathcal{S}}(N, \mathbf{Z})$ are the kernel of the boundary map $\partial(\{\alpha, \beta\}) = \{\beta\} - \{\alpha\}$:

$$0 \to \underline{\mathcal{S}}(N, \mathbf{Z}) \to \underline{\mathcal{M}}(N, \mathbf{Z}) \xrightarrow{\partial} \underline{\mathcal{B}}(N, \mathbf{Z}).$$

The Hecke algebra act on $\underline{\mathcal{M}}(N, \mathbf{Z})$ and there is an involution $*\{\alpha, \beta\} = \{-\alpha, -\beta\}$. Integration defines a pairing

$$S_2(N) \times \underline{\mathcal{M}}(N, \mathbf{Z}) \to \mathbf{C}.$$

The Manin-Drinfeld theorem asserts that the image of $\underline{\mathcal{M}}(N, \mathbf{Z})$ in $\mathrm{Hom}(S_2(N), \mathbf{C})$ is a lattice. There is a natural isomorphism between $\underline{\mathcal{S}}(N, \mathbf{Z})$ and $H_1(X_0(N), \mathbf{Z})$.

2.2 The Method of Graphs

We briefly review the method of graphs, see [13] and [15] for more details. Let M be a positive integer, p a prime not dividing M, and put $N = pM$. Let D be the finitely generated free abelian group on the superingular points of $X_0(M)(\overline{\mathbf{F}}_p)$, i.e., the enhanced elliptic curves $\mathbf{E} = (E, C)$ where E is a supersingular elliptic curve defined over $\overline{\mathbf{F}}_p$ and C is a cyclic subgroup of order M, and enhanced curves are identified if they are isomorphic in the evident way. Let $w_{\mathbf{E}} = \frac{|\,\mathrm{Aut}(\mathbf{E})|}{2}$ where $\mathrm{Aut}(\mathbf{E})$ is the group of $\overline{\mathbf{F}}_p$-automorphisms of $\mathbf{E}$. We have $w_{\mathbf{E}} \leq 12$ and if $p \geq 5$ then $w_{\mathbf{E}} \leq 3$. The **monodromy pairing** on D is

$$\langle \mathbf{E}, \mathbf{E}' \rangle = \begin{cases} w_{\mathbf{E}} & \text{if } \mathbf{E} = \mathbf{E}' \\ 0 & \text{if } \mathbf{E} \neq \mathbf{E}' \end{cases}$$

The Hecke operators act on D in a way compatible with this pairing. Define

$$X_{N,p} = \{\sum a_{\mathbf{E}} \mathbf{E} : \sum a_{\mathbf{E}} = 0\}$$

It is known that $X_{N,p} \otimes \mathbf{C}$ is isomorphic as a Hecke module to the subspace of $S_2(N, \mathbf{C})$ generated by newforms and oldforms of level pd for $d|M$.

David Kohel [8] has implimented an algorithm which computes the action of the Hecke operators on $X_{N,p}$ using the arithmetic of quaternion algebras.

2.3 Enumerating quotients of $J_0(N)$

It is necessary to list all newforms of a given level N. This can be done by decomposing the new subspace of the modular symbols $\underline{\mathcal{S}}(N, \mathbf{Q})$ using the Hecke operators. The characteristic polynomial of T_2 is computed, and then T_2 is used to break up the space. The process is applied recursively with $T_3, T_5, \ldots$ until it terminates. After computing the decomposition we order the newforms as suggested by Cremona: First by dimension. Within each dimension, in binary by the signs of the Atkin-Lehner involutions, e.g., $+++$, $++-$, $+-+$, $+--$, $-++$, etc. When two forms have the same involutions, order by $|\operatorname{Tr}(a_p)|$ with ties broken by taking the positive trace first. For historical reasons this does not always agree with the ordering in Cremona's tables (page 5 of [3]). There is only one case in our table in which the two ordering schemes disagree, our **446B** is Cremona's **446D**.

2.4 The Modular Polarization

A polarization of an abelian variety A is an isogeny between A and its dual arising from a very amply invertible sheaf (see [14]). $J_0(N)$ is a Jacobian so it possesses a canonical polarization arising from the θ-divisor and this induces the **modular polarization** $\theta_f : A_f^\vee \to A_f$.

$$\begin{tikzcd} A_f^\vee \arrow[r] \arrow[dr, "\theta_f"] & J_0(N) \arrow[d, two heads] \\ & A_f \end{tikzcd}$$

By ([14], Theorem 13.3) $\deg(\theta_f)$ is a perfect square so we may define the **modular degree** $\delta_f = \sqrt{\deg(\theta_f)}$. The kernel of θ_f is the intersection of $A_f^\vee$ with $I_f J_0(N)$ so it measure intersections between $A_f^\vee$ and other factors of $J_0(N)$.

Proposition 2.1. *With notation as in Theorem 1.1,*

$$\operatorname{Ker}(\theta_f) \cong \operatorname{Coker}(H[I_f] \to \Phi_f(H))$$

Proof. Delete the middle column of the diagram in Theorem 1.1 and apply the snake lemma. $\square$

Using Lemma 2.5, to be proved later, we see that the modular degree can be computed as follows: Let $\varphi_1, \dots, \varphi_{2d}$ be a basis for $\operatorname{Hom}(H, \mathbf{Z})[I_f]$ and $a_1, \dots, a_{2d}$ a basis for $H[I_f]$. Then δ_f is the square root of the absolute value of the determinant of the matrix $(\varphi_i(a_j))$.

2.5 Torsion

We can obtain both upper and lower bounds on $A_f(\mathbf{Q})_{\mathrm{tor}}$ and $A_f^\vee(\mathbf{Q})_{\mathrm{tor}}$. For the examples considered in our table these bounds were sufficient to determine the odd parts of these groups. Let $\chi_p(X) \in \mathbf{Z}[X]$ denote the characteristic polynomial of T_p acting on A_f. It is a polynomial having integer coefficients and degree equal to $\dim A_f$.

Proposition 2.2. *Both* $|A_f(\mathbf{Q})_{\mathrm{tor}}|$ *and* $|A_f^\vee(\mathbf{Q})_{\mathrm{tor}}|$ *divide*

$$\gcd\{\chi_p(p+1) : (p, 2N) = 1,\ p \text{ prime}\}.$$

Proof. Use the Eichler-Shimura relation and injectivity of rational torsion under reduction modulo an odd prime p (since $1 = e < p - 1$). □

The difference of cusps $\alpha, \beta \in X_0(N)$ define a point $(\alpha) - (\beta) \in J_0(N)(\mathbf{C})$.

Proposition 2.3. *The order of the image of* $(\alpha) - (\beta)$ *in* $A_f(\mathbf{C})$ *equals the order of the image of the modular symbol* $\{\alpha, \beta\}$ *in*

$$\frac{\Phi_f(\underline{\mathcal{M}}(N, \mathbf{Z}))}{\Phi_f(\underline{\mathcal{S}}(N, \mathbf{Z}))}.$$

Proof. By the classical Abel-Jacobi theorem ([10], ch IV, Theorem 2.2), the modular symbol $\{\alpha, \beta\}$ maps, via the period map, to the point $(\alpha) - (\beta) \in J_0(N)(\mathbf{C})$. Now use Theorem 1.1. □

In particular, the point $(0) - (\infty) \in J_0(N)(\mathbf{Q})$ generates a cyclic subgroup of $A_f(\mathbf{Q})$ and this gives a lower bound on $A_f(\mathbf{Q})_{\mathrm{tor}}$.

2.6 Component Groups

Let p be a prime exactly dividing N and $\Phi_{A,p}$ the component group of $A = A_f$. Thus we have the exact sequence

$$0 \to \mathcal{A}^0_{\mathbf{F}_p} \to \mathcal{A}_{\mathbf{F}_p} \to \Phi_{A,p} \to 0$$

with $\mathcal{A}$ the Neron model of A and $\mathcal{A}^0_{\mathbf{F}_p}$ connected. Let $X[I_f]$ be the submodule of $X = X_{N,p}$ cut out by the annihilator of f. The monodromy pairing defines a map $X \to \mathrm{Hom}(X[I_f], \mathbf{Z})$. Let δ_f be the modular degree and w_p the sign of the Atkin-Lehner involution W_p on f. The following will be proved in [18].

Theorem 2.4.

$$\begin{aligned} |\Phi_{A,p}(\overline{\mathbf{F}}_p)| &= \delta_f \cdot \frac{|\operatorname{Coker}(X \to \mathrm{Hom}(X[I_f], \mathbf{Z}))|^2}{\operatorname{Disc}(X[I_f] \times X[I_f] \to \mathbf{Z})}, \\ |\Phi_{A,p}(\mathbf{F}_p)| &= \begin{cases} |\Phi_{A,p}(\overline{\mathbf{F}}_p)| & w_q = -1 \\ |\Phi_{A,p}(\overline{\mathbf{F}}_p)[2]| & w_q = +1 \end{cases} \end{aligned}$$

2.7 Rational part of the special value

Let $\underline{\mathcal{M}}(\mathbf{Q}) = \underline{\mathcal{M}}(N, \mathbf{Q})$ and extend Φ_f to a map $\underline{\mathcal{M}}(\mathbf{Q}) \to \mathbf{C}$. Then Φ_f has a rational structure in the following sense.

Lemma 2.5. *Let* $\varphi_1, \ldots, \varphi_n$ *be a* $\mathbf{Q}$*-basis for* $\mathrm{Hom}(\underline{\mathcal{M}}(\mathbf{Q}), \mathbf{Q})[I_f]$ *and set*

$$\Psi = \varphi_1 \times \cdots \times \varphi_n : \underline{\mathcal{M}}(\mathbf{Q}) \to \mathbf{Q}^n.$$

Then $n = 2d$ *and* $\mathrm{Ker}(\Psi) = \mathrm{Ker}(\Phi_f)$.

Proof. This result is due to Shimura ([17]), but we sketch a proof. To compute $\dim \mathrm{Hom}(\underline{\mathcal{M}}(\mathbf{Q}), \mathbf{Q})[I_f]$ we may first tensor with $\mathbf{C}$. Let $\overline{S}_2$ denote the weight 2 anti-holomorphic cusp forms and E_2 the weight 2 Eisenstein series for $\Gamma_0(N)$. Then $\underline{\mathcal{M}}(\mathbf{C})$ is isomorphic as a $\mathbf{T}$-module to $S_2 \oplus \overline{S}_2 \oplus E_2$ (prop. 9 of [12] and the Eichler-Shimura embedding). Because of the Peterson inner product, the dual $\mathrm{Hom}(\underline{\mathcal{M}}(\mathbf{C}), \mathbf{C})$ is also isomorphic as a $\mathbf{T}$-module to $S_2 \oplus \overline{S}_2 \oplus E_2$. Since f is new, by the Atkin-Lehner multiplicity one theory,

$$(S_2 \oplus \overline{S}_2 \oplus E_2)[I_f] = S_2[I_f] \oplus \overline{S}_2[I_f]$$

has complex dimension $2d$, which gives the first assertion.

Next note that $\mathrm{Ker}(\Phi_f) \otimes \mathbf{C} \subset \mathrm{Ker}(\Psi) \otimes \mathbf{C}$ because each map $x \mapsto \langle f_i, x\rangle$ lies in $\mathrm{Hom}(\underline{\mathcal{M}}(\mathbf{Q}), \mathbf{C})[I_f]$ and $\mathrm{Ker}(\Psi) \otimes \mathbf{C}$ is the intersection of the kernels of *all* maps in $\mathrm{Hom}(\underline{\mathcal{M}}(\mathbf{Q}), \mathbf{C})[I_f]$. By Theorem 1.1 the image of Φ_f is a lattice, so $\dim_{\mathbf{Q}} \mathrm{Ker}(\Phi_f) = \dim_{\mathbf{Q}} \underline{\mathcal{M}}(\mathbf{Q}) - 2d$. Since Ψ is the intersection of the kernels of $n = 2d$ independent linear functionals $\varphi_1, \dots, \varphi_n$, $\mathrm{Ker}(\Psi)$ also has dimension $\dim \underline{\mathcal{M}}(\mathbf{Q}) - 2d$. Since the dimensions are the same and there is an inclusion, we have an equality $\mathrm{Ker}(\Phi_f) \otimes \mathbf{C} = \mathrm{Ker}(\Psi) \otimes \mathbf{C}$ which forces $\mathrm{Ker}(\Phi_f) = \mathrm{Ker}(\Psi)$. □

Let V be a finite dimensional vector space over $\mathbf{R}$. A **lattice** $L \subset V$ is a free abelian group of rank $= \dim V$ such that $\mathbf{R}L = V$. If $L, M \subset V$ are lattices, the **lattice index** $[L : M]$ is the absolute value of the determinant of an automorphism of V taking L isomorphically onto M. Extend the definition to the case when M has rank strictly smaller than $\dim V$ by defining $[L : M] = 0$.

Lemma 2.6. *Suppose $\tau_i : V \to W_i$, $i = 1, 2$ are surjective linear maps such that $\mathrm{Ker}(\tau_1) = \mathrm{Ker}(\tau_2)$. Then*

$$[\tau_1(L) : \tau_1(M)] = [\tau_2(L) : \tau_2(M)].$$

Proof. Surjectivety and equality of kernels insures that there is a unique isomorphism $\iota : W_1 \to W_2$ such that $\iota\tau_1 = \tau_2$. Let σ be an automorphism of W_1 such that $\sigma(\tau_1(L)) = \tau_1(M)$. Then

$$\iota\sigma\iota^{-1}(\tau_2(L)) = \iota\sigma\tau_1(L) = \iota\tau_1(M) = \tau_2(M).$$

Since conjugation doesn't change the determinant,

$$[\tau_2(L) : \tau_2(M)] = |\det(\iota\sigma\iota^{-1})| = |\det(\sigma)| = [\tau_1(L) : \tau_1(M)].$$

□

Let $S_2(N, \mathbf{Z})$ be the space of cusp forms whose q-expansion at infinity hass integer coefficients. Let Ω_f^0 be the measure of the identity component of $A_f(\mathbf{R})$ with respect to an integral basis for $S_f(\mathbf{Z}) = S_2(N, \mathbf{Z})[I_f]$. Let $\mathbf{e} = \{0, \infty\} \in \underline{\mathcal{M}}(N, \mathbf{Z})$ denote the **winding element**.

Theorem 2.7. *Let Ψ be as in Lemma 2.5. Then*

$$\pm\frac{L(A_f, 1)}{\Omega_f^0} = [\Psi(\underline{\mathcal{S}}(N, \mathbf{Z})^+) : \Psi(\mathbf{T}\mathbf{e})]$$

Proof. Let $\Phi = \Phi_f$ be the period map defined by a basis $f_1, \ldots, f_d$ of conjugate newforms. The image of Φ, which we identify with $\mathbf{C}^d$, is an algebra with unit element $\mathbf{1} = (1, \ldots, 1)$ equipped with an action of the Hecke operators: T_p acts as $(a_p^{(1)}, \ldots, a_p^{(d)})$ where the components are the Galois conjugates of a_p. Let $\mathbf{Z}^d \subset \mathbf{R}^d \subset \mathbf{C}^d$ be the usual submodules. Let $\mathrm{Vol}(\underline{\mathcal{S}}^+)$ be the volume of the image of $\underline{\mathcal{S}}^+ = \underline{\mathcal{S}}(N, \mathbf{Z})^+$ under Φ. Observe that $\mathrm{Vol}(\underline{\mathcal{S}}^+) = [\mathbf{Z}^d : \Phi(\underline{\mathcal{S}}^+)]$ and $|L(A_f, 1)| = [\mathbf{Z}^d : \Phi(\mathbf{e})\mathbf{Z}^d]$. Let $W \subset \mathbf{C}^d$ be the $\mathbf{Z}$-module spanned by the columns of a basis for $S_f(\mathbf{Z})$. Because Ω_f^0 is computed with respect to a basis for $S_f(\mathbf{Z})$,

$$\mathrm{Vol}(\underline{\mathcal{S}}^+) = [W : \mathbf{T1}] \cdot \Omega_f^0.$$

Because $S_2(N, \mathbf{Z})$ is saturated, $[\mathbf{Z}^d : W] = 1$ so $[\mathbf{Z}^d : \mathbf{T1}] = [W : \mathbf{T1}]$. The following calculation involves lattices in $\mathbf{R}^d$:

$$\begin{aligned}
[\Phi(\underline{\mathcal{S}}^+) : \Phi(\mathbf{Te})] &= [\Phi(\underline{\mathcal{S}}^+) : \mathbf{Z}^d] \cdot [\mathbf{Z}^d : \Phi(\mathbf{Te})] \\
&= \frac{1}{[\mathbf{Z}^d : \Phi(\underline{\mathcal{S}}^+)]} \cdot [\mathbf{Z}^d : \Phi(\mathbf{Te})] \\
&= \frac{1}{\mathrm{Vol}(\underline{\mathcal{S}}^+)} \cdot [\mathbf{Z}^d : \Phi(\mathbf{e})\mathbf{Z}^d] \cdot [\Phi(\mathbf{e})\mathbf{Z}^d : \Phi(\mathbf{Te})] \\
&= \frac{|L(A_f, 1)|}{\mathrm{Vol}(\underline{\mathcal{S}}^+)} \cdot [\Phi(\mathbf{e})\mathbf{Z}^d : \Phi(\mathbf{Te})] \\
&= \frac{|L(A_f, 1)|}{\mathrm{Vol}(\underline{\mathcal{S}}^+)} \cdot [\Phi(\mathbf{e})\mathbf{Z}^d : \Phi(\mathbf{e})\mathbf{T1}] \\
&= \frac{|L(A_f, 1)|}{\Omega_f^0 \cdot [W : \mathbf{T1}]} \cdot [\mathbf{Z}^d : \mathbf{T1}] \\
&= \frac{|L(A_f, 1)|}{\Omega_f^0}.
\end{aligned}$$

The theorem now follows from lemmas 2.5, 2.6, and the fact that f has real Fourier coefficients so $L(A_f, 1) \in \mathbf{R}$ hence $|L(A_f, 1)| = \pm L(A_f, 1)$.

$\square$

Corollary 2.8. *Let n_f be the order of the image in $A_f(\mathbf{Q})$ of the point $(0) - (\infty) \in J_0(N)(\mathbf{Q})$. Then*

$$\frac{L(A_f, 1)}{\Omega_f^0} \in \frac{1}{n_f}\mathbf{Z}.$$

Proof. Let x denote the image of $(0) - (\infty) \in A_f(\mathbf{Q})$ and set $I = \mathrm{Ann}(x) \subset \mathbf{T}$. Since f is a *newform* the Hecke operators T_p for $p|N$ act as 0 or ± 1 on $A_f(\mathbf{Q})$ (end of section 6 of [6]). If $p \nmid N$ a standard calculation (section 2.8 of [3]) combined with the Abel-Jacobi theorem shows that $T_p(x) = (p+1)x$. Let $C = \mathbf{Z}x$ denote the (finite, by Manin-Drinfeld) cyclic subgroup of $A_f(\mathbf{Q})$ generated by x, so n_f is the order of C. There is an injection $\mathbf{T}/I \hookrightarrow C$ sending T_p to

$T_p(x)$. By the theorem, we have

$$\begin{aligned}
\pm L(A_f,1)/\Omega_f^0 &= [\Psi(\underline{\mathcal{S}}^+) : \Psi(\mathbf{T}e)] \\
&= [\Psi(\underline{\mathcal{S}}^+) : \Psi(I\mathbf{e})] \cdot [\Psi(I\mathbf{e}) : \Psi(\mathbf{T}\mathbf{e})] \\
&= [\Psi(\underline{\mathcal{S}}^+) : I\Psi(\mathbf{e})] \cdot [I\Psi(\mathbf{e}) : \mathbf{T}\Psi(\mathbf{e})] \\
&= \frac{[\Psi(\underline{\mathcal{S}}^+) : I\Psi(\mathbf{e})]}{[\mathbf{T}\Psi(\mathbf{e}) : I\Psi(\mathbf{e})]} \in \frac{1}{n_f}\mathbf{Z}.
\end{aligned}$$

The final inclusion follows from two observations. By Abel-Jacobi, I is exactly those elements of $\mathbf{T}$ which send $\Psi(\mathbf{e})$ into $\Psi(\underline{\mathcal{S}}^+)$, so $[\Psi(\underline{\mathcal{S}}^+) : I\Psi(\mathbf{e})] \in \mathbf{Z}$. Second, there is a surjective map

$$\mathbf{T}/I \to \frac{\mathbf{T}\Psi(\mathbf{e})}{I\Psi(\mathbf{e})}$$

sending t to $t\Psi(\mathbf{e})$, so $[\mathbf{T}\Psi(\mathbf{e}) : I\Psi(\mathbf{e})]$ divides $n_f = |C| = |\mathbf{T}/I|$. $\square$

2.8 Intersections

Let f, g be nonconjugate newforms and $H = H_1(X_0(N), \mathbf{Z})$.

Proposition 2.9. $(A_f^\vee \cap A_g^\vee)[p] \neq 0$ *iff the mod p rank of* $H[I_f] + H[I_g]$ *is strictly less than* $\operatorname{rank} H[I_f] + \operatorname{rank} H[I_g]$.

Proof. By (1.1) $\Lambda_f = H[I_f]$ (resp., $\Lambda_g = H[I_g]$) is the submodule of H which defines A_f (resp., A_g). By reduction mod p we mean the map $H \to H \otimes \mathbf{F}_p$. Suppose

$$\operatorname{rank}(\Lambda_f + \Lambda_g) \bmod p < \operatorname{rank}\Lambda_f + \operatorname{rank}\Lambda_g.$$

Since Λ_f (resp., Λ_g) is a kernel, it is saturated, so $\operatorname{rank}\Lambda_f \bmod p = \operatorname{rank}\Lambda_f$ (resp., for Λ_g). We conclude that the mod p linear dependence must involve vectors from both Λ_f and Λ_g; there is $v \in \Lambda_f$ and $w \in \Lambda_g$ so that $v, w \not\equiv 0 \bmod p$ but $v + w \equiv 0 \bmod p$. Thus $\frac{v+w}{p} \in H$ is integral, i.e., in $J_0(N)(\mathbf{C})$ we have $\frac{1}{p}v - (-\frac{1}{p}w) = 0$. But $\frac{1}{p}v \notin \Lambda_f$ and $\frac{1}{p}w \notin \Lambda_g$ (otherwise v and w would be $0 \bmod p$), so $\frac{1}{p}v$ and $-\frac{1}{p}w$ are both nontrivial p-torsion in $A_f^\vee$, $A_g^\vee$, resp. Conclusion: $0 \neq \frac{1}{p}v = -\frac{1}{p}w \in (A_f^\vee \cap A_g^\vee)[p]$.

Conversely, suppose $0 \neq x \in (A_f^\vee \cap A_g^\vee)[p]$. Choose lifts modulo H to $x_f \in \frac{1}{p}\Lambda_f$ and $x_g \in \frac{1}{p}\Lambda_g$. Then $px_f \in \Lambda_f$ (resp., $px_g \in \Lambda_g$), but $px_f \notin pH$ (resp., $px_g \notin pH$) because $x \neq 0$. Since $x_f - x_g \in H$, $px_f - px_g = p(x_f - x_g) \equiv 0 \bmod p$. This is a nontrivial linear relation between Λ_f and Λ_g. $\square$

Corollary 2.10. *If $p > 2$ and the sign of some Atkin-Lehner involution for f is different than that for g then* $(A_f^\vee \cap A_g^\vee)[p] = 0$.

Proof. Suppose $w_q(f) \neq w_q(g)$ and let $G = (A_f^\vee \cap A_g^\vee)[p]$. Observe that W_q acts as $w_q(f) \bmod p$ on $A_f^\vee[p]$ and as $w_q(g) \bmod p$ on $A_g^\vee[p]$. Hence W_q acts as both $w_q(f) \bmod p$ and $w_q(g) \bmod p$ on G. Since $p > 2$, this is not possible when $G \neq 0$. $\square$

3 Results

This section contains tables computed using the above algorithms as implimented in the author's program `HECKE` (a `C++` program using `LiDIA` and `NTL`), David Kohel's `Magma` software, and `PARI`. Each factor A_f of $J_0(N)$ is denoted as follows:

N isogeny-class dimension

The dimension frequently determines the factor, so it is included in the notation. We consider only the odd part of Ш so we only computed the odd parts of the arithmetic invariants of A_f. Thus at this point we make the

WARNING: ONLY ODD PARTS OF INVARIANTS ARE GIVEN!

Tables 1-3: New Visible Ш
Let n_f be the largest odd square dividing the numerator of $L(A_f, 1)/\Omega_f$. Table 1 lists those A_f such that, for $p|n_f$ there exists a *new* factor B_g of $J_0(N)$, of positive analytic rank, and such that $(A_f^\vee \cap B_g^\vee)[p] \neq 0$. This is necessary (and usually sufficient) for the p-torsion in the new visible part of Ш to be nonzero. In many cases it could be seen that there were no other appropriate new factors by looking at the signs of the Atkin-Lehner involutions. Up to level 1001 our search was systematic. The two examples after level 1001 were not found by systematic search (i.e., there may be a gap). In those cases for which $4|N$, we put $c_2 = a$, as we don't know how to compute c_2 exactly when the reduction is additive. Table 2 contains further arithmetic information about each explanatory factor.

The explanatory factors of level ≤ 1028 are *exactly* the set of rank 2 elliptic curves of level ≤ 1028. By [2], the explanatory factor at level 1061 is the first surface of rank 4 (and prime level).

Table 4: Component groups
Table 4 gives the quantities involved in the formula for Tamagawa numbers, for each of the A_f from table 1.

Table 5: Odd square numerator
In order to find the A_f, we first enumerated those A_f for which the numerator of $L(A_f, 1)/\Omega_f$ is divisible by an odd square n_f. For $N < 1000$, these are given in table 5. Any odd visible Ш coprime to primes dividing torsion and c_p must show up as a divisor of the numerator, and given BSD, it must show up as a square divisor because the Mordell-Weil rank of the explanatory factor is even. It would be interesting to compute the conjectural order of Ш for each abelian variety in this table, but not in table 1, and show (when possible) that the visible Ш is old.

$\mathbf{A_f}$	n_f	w_q	c_p	T	$TL(1)/\Omega_f$	δ_A	$\mathbf{B_g}$
389E20	5^2	$-$	97	97	5^2	5	**389A1**
433D16	7^2	$-$	3^2	3^2	7^2	$3 \cdot 7 \cdot 37$	**433A1**
446F8	11^2	$+-$	$1, 3$	3	11^2	$11 \cdot 359353$	**446B1**
563E31	13^2	$-$	281	281	13^2	13	**563A1**
571D2	3^2	$-$	1	1	3^2	$3^2 \cdot 127$	**571B1**
655D13	3^4	$+-$	$1, 1$	1	3^4	$3^2 \cdot 19 \cdot 515741$	**655A1**
664F8	5^2	$-+$	$a, 1$	1	5^2	5	**664A1**
681B1	3^2	$+-$	$1, 1$	1	3^2	$3 \cdot 5^3$	**681C1**
707G15	13^2	$+-$	$1, 1$	1	13^2	$13 \cdot 800077$	**707A1**
709C30	11^2	$-$	59	59	11^2	11	**709A1**
718F7	7^2	$+-$	$1, 1$	1	7^2	$7 \cdot 151 \cdot 35573$	**718B1**
794G14	11^2	$+-$	$3, 1$	3	11^2	$3 \cdot 7 \cdot 11 \cdot 47 \cdot 35447$	**794A1**
817E15	7^2	$+-$	$1, 5$	5	7^2	$7 \cdot 79$	**817A1**
916G9	11^2	$-+$	$a, 1$	1	11^2	$3^9 \cdot 11 \cdot 17 \cdot 239$	**916C1**
944O6	7^2	$+-$	$a, 1$	1	7^2	7	**944E1**
997H42	3^4	$-$	83	83	3^4	3^2	**997B1**, **C1**
1001L7	7^2	$+-+$	$1, 1, 1$	1	7^2	$7 \cdot 19 \cdot 47 \cdot 2273$	**1001C1**
1028E14	$3^2 \cdot 11^2$	$-+$	$a, 1$	3	$3^4 \cdot 11^2$	$3^{13} \cdot 11$	**1028A1**
1061D46	151^2	$-$	$5 \cdot 53$	$5 \cdot 53$	151^2	$61 \cdot 151 \cdot 179$	**1061B1**

Table 1: New visible III

$\mathbf{B_g}$	rank	w_q	c_p	T	δ_B	Comments
389A1	2	$-$	1	1	5	first curve of rank 2
433A1	2	$-$	1	1	7	
446B1	2	$+-$	$1, 1$	1	11	this is **446D** in [3]
563A1	2	$-$	1	1	13	
571B1	2	$-$	1	1	3	
655A1	2	$+-$	$1, 1$	1	3^2	
664A1	2	$-+$	$1, 1$	1	5	
681C1	2	$+-$	$1, 1$	1	3	
707A1	2	$+-$	$1, 1$	1	13	
709A1	2	$-$	1	1	11	
718B1	2	$+-$	$1, 1$	1	7	
794A1	2	$+-$	$1, 1$	1	11	
817A1	2	$+-$	$1, 1$	1	7	
916C1	2	$-+$	$3, 1$	1	$3 \cdot 11$	
944E1	2	$+-$	$1, 1$	1	7	
997B1	2	$-$	1	1	3	
997C1	2	$-$	1	1	3	
1001C1	2	$+-+$	$1, 3, 1$	1	$3^2 \cdot 7$	
1028A1	2	$-+$	$3, 1$	1	$3 \cdot 11$	intersects **1028E** mod 11
1061B2	4	$-$	1	1	151	first surface of rank 4 [2]

Table 2: Explanatory factors

446 $= 2 \cdot 223$	**655** $= 5 \cdot 131$	**664** $= 2^3 \cdot 83$	**681** $= 3 \cdot 227$
707 $= 7 \cdot 101$	**718** $= 2 \cdot 359$	**794** $= 2 \cdot 397$	**817** $= 19 \cdot 43$
916 $= 2^2 \cdot 229$	**944** $= 2^4 \cdot 59$	**1001** $= 7 \cdot 11 \cdot 13$	**1028** $= 2^2 \cdot 257$

Table 3: Factorizations

$\mathbf{A_f}$	p	w_p	$\lvert\,\mathrm{Coker}\,\rvert$	$\lvert\,\mathrm{Disc}(X[I_f])\rvert$	$\lvert\Phi(\overline{\mathbf{F}}_p)\rvert$
389E20	389	$-$	97	$5\cdot 97$	97
433D16	433	$-$	3^2	$3^3\cdot 7\cdot 37$	3^2
446F8	223	$-$	3	$3\cdot 11\cdot 359353$	3
	2	$+$	3	$3\cdot 11$	$3\cdot 359353$
563E31	563	$-$	281	$13\cdot 281$	281
571D2	571	$-$	1	$3^2\cdot 127$	1
655D13	131	$-$	1	$3^2\cdot 19\cdot 515741$	1
	5	$+$	1	3^2	$19\cdot 515741$
664F8	83	$+$	1	5	1
681B1	227	$-$	1	$3\cdot 5^3$	1
	3	$+$	1	$3\cdot 5^2$	5
707G15	101	$-$	1	$13\cdot 800077$	1
	7	$+$	1	13	800077
709C30	709	$-$	59	$11\cdot 59$	59
718F7	359	$-$	1	$7\cdot 151\cdot 35573$	1
	2	$+$	1	7	$151\cdot 35573$
794G14	397	$-$	3	$3^2\cdot 7\cdot 11\cdot 47\cdot 35447$	3
	2	$+$	3	$3\cdot 11$	$3^2\cdot 7\cdot 47\cdot 35447$
817E15	43	$-$	5	$5\cdot 7\cdot 79$	5
	19	$+$	1	7	79
916G9	229	$+$	1	$3^9\cdot 11\cdot 17\cdot 239$	1
944O6	59	$-$	1	7	1
997H42	997	$-$	83	$3^2\cdot 83$	83
1001L7	13	$+$	1	$7\cdot 19\cdot 47\cdot 2273$	1
	11	$-$	1	$7\cdot 19\cdot 47\cdot 2273$	1
	7	$+$	1	$7\cdot 19\cdot 47$	2273
1028E14	257	$+$	1	$3^{13}\cdot 11$	1
1061D46	1061	$-$	$5\cdot 53$	$5\cdot 53\cdot 61\cdot 151\cdot 179$	$5\cdot 53$

Table 4: Component groups

305D7 : 3	**309D8** : 5	**335E11** : 3^2	**389E20** : 5	**394A2** : 5	**399G5** : 3^4
433D16 : 7	**435G2** : 3	**436C4** : 3	**446E7** : 3	**446F8** : 11	**455D4** : 3
473F9 : 3	**500C4** : 3	**502E6** : 11	**506I4** : 5	**524D4** : 3	**530G4** : 7
538E7 : 3	**551H18** : 3	**553D13** : 3	**555E2** : 3	**556C7** : 3	**563E31** : 13
564C3 : 3	**571D2** : 3	**579G13** : $3\cdot 5$	**597E14** : 19	**602G3** : 3	**604C6** : 3
615F6 : 5	**615G8** : 7	**620D3** : 3	**620E4** : 3	**626F12** : 5	**629G15** : 3
642D2 : 3	**644C5** : 3	**644D5** : 3	**655D13** : 3^2	**660F2** : 3	**662E10** : 43
664F8 : 5	**668B5** : 3	**678I2** : 3	**681B1** : 3	**681I10** : 3	**682I6** : 11
707G15 : 13	**709C30** : 11	**718F7** : 7	**721F14** : 3^2	**724C8** : 3	**756G2** : 3
764A8 : 3	**765M4** : 3	**766B4** : 3	**772C9** : 3	**790H6** : 3	**794G12** : 11
794H14 : 5^2	**796C8** : 3	**817E15** : 7	**820C4** : 3	**825E2** : 3	**844C10** : 3^2
855M4 : 3	**860D4** : 3	**868E5** : 3	**876E5** : 3	**878C2** : 3	**884D6** : 3
885L9 : 3^2	**894H2** : 3	**902I5** : 3	**913G17** : 3	**916G9** : 11	**918O2** : 5
918P2 : 3	**925K7** : 3	**932B13** : 3^2	**933E14** : 19	**934I12** : 7	**944O6** : 7
946K7 : 3	**949B2** : 3	**951D19** : 3	**959D24** : 3	**964C12** : 3^2	**966J1** : 3
970I5 : 3	**980F1** : 3	**980J2** : 3	**986J7** : 5	**989E22** : 5	**993B3** : 3^2
996E4 : 3	**997H42** : 3^2	**998A2** : 3	**998H9** : 3	**999J10** : 3	

Table 5: Odd square numerator

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