Arizona Winter School lectures

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LECTURE 4

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4.1 The family

$$X : t(X_0^5 + \cdots + X_4^5) = X_0 \cdots X_4$$
$$f \downarrow$$
$$S = \mathbb{P}^1(\mathbb{C}) \setminus \{0, \tfrac{\mu_5(\mathbb{C})}{5}\}$$

This is a family of CY three folds with

$$H^3(X_t, \mathbb{Z}) \cong \mathbb{Z}^{204}$$

Consider the group

$$G = \{(\zeta_0, \ldots, \zeta_4) \in \mu_5^5 \mid \prod \zeta_i = 1\}$$

and take

$$V_{\mathbb{Z}} = \left(R^3 f_*(\underline{\mathbb{Z}})\right)^G$$

Proposition 4.1.1

(a) The rank of $V_{\mathbb{Z}}$ is 4, and the weight 3 Hodge numbers are $h^{3,0} = h^{2,1} = h^{1,2} = h^{0,3} = 1$.

(b) The variation is not constant

4.1.2 Another geometric realization

Note that

$$V_{t,\mathbb{Q}} = H^3(X_t, \mathbb{Q})^G = H^3(X_t/G, \mathbb{Q}).$$

To compute X_t/G note

$$\mathbb{C}[X_0, \ldots, X_4]^G = \mathbb{C}[Y_0, \ldots, Y_5] / (Y_5^5 - Y_0 \cdots Y_4)$$

where

$$Y_i = X_i^5 \qquad i = 0, \ldots, 4$$

and

$$Y_5 = X_0 \cdots X_4$$

Thus

$$X_t/G = \operatorname{Proj} \mathbb{C}[Y_0, \ldots, Y_5] / \left(Y_5^5 - Y_0 \cdots Y_4,\ t(Y_0 + \cdots + Y_4) = Y_5 \right).$$

$$= \operatorname{Proj} \left(\mathbb{C}[Y_0 \cdots Y_4] / t^5 (Y_0 + \cdots + Y_4)^5 = Y_0 \cdots Y_4 \right)$$

For a parameter $s \in \overset{S_{new} =}{\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1/5^5, \infty\}}$ set

$$Y_s : \quad s(Y_0 + \dots + Y_4)^5 = Y_0 \cdots Y_4$$

Then

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow f_{new} \\ S & \xrightarrow{\pi} & S_{new} \\ t & \longmapsto & s = t^5 \end{array}$$

We see

$$V_{\mathbb{Q}} = \pi^* R^3 f_{new *} \mathbb{Q}$$

and the betti #'s of Y_s are

1, 0, 1, 4, 1, 0, 1

"The only interesting part of $H^*(Y_s)$ is the H^3 which is $V_{s^{1/5}, \mathbb{Q}}$"

4.2 A thesis problem

4.2.1 For how many $t \in S$ is the Hodge structure $V_{t,\mathbb{Z}}$ reducible?

Reducible means:

$$V_{t,\mathbb{Q}} = V_1 \oplus V_2 \quad \text{over } \mathbb{Q}$$

$$V_1 \otimes \mathbb{C} = V_t^{3,0} \oplus V_t^{0,3} \text{ and } V_2 \otimes \mathbb{C} = V_t^{2,1} \oplus V_t^{1,2}$$

Remarks 4.2.2 (Unexplained)

(a) This is not the same as asking the H.S. to be "CM" (means MT = Torus). Prof. Oort & me expect that CM happens only for a finite # of t's

(b) For very general $t \in S$ the HS ~~is~~ probably neither CM nor reducible.

(c) The generalized Hodge conjecture

implies

$V_{t,\mathbb{Z}}$ reducible $\Longleftrightarrow$ $\exists$ algebraic family of ~~one~~ 1-cycles $C_\lambda \subset X_t$, $\lambda \in \Lambda$ such that

$$\begin{tikzcd} AJ : \Lambda \arrow[r] \arrow[rr, bend right] & J^3(X_t) \arrow[r] & J^3(X_t)^G \end{tikzcd}$$

is not constant

In particular we see that such $t \in \overline{\mathbb{Q}}$.

(d) The implication

$$V_{t,\mathbb{Z}} \text{ reducible} \Longrightarrow t \in \overline{\mathbb{Q}}$$

also follows from "$H \Rightarrow AH$".

(e) Assuming "$H \Rightarrow AH$" one can also show

$V_{t,\mathbb{Z}}$ red $\Rightarrow$ $1/t \in \overline{\mathbb{Z}}[1/5]$ (not so easy).

4.3 PVHS of wt 3 and type (1,1,1,1)

$$(V_{\mathbb{Z}}, V_s^{p,q}, \psi) / D$$

Lemma 4.3.1 There exist many bases e_1, e_2, e_3, e_4 of $V_{\mathbb{Q}}$ s.t.

$$\psi \sim \frac{1}{(2\pi i)^3} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

For a suitable choice of such basis and possibly shrinking D we will have

$$F_s^3 = V_s^{3,0} = \mathbb{C} \cdot \begin{pmatrix} 1 \\ a \\ b \\ c \end{pmatrix}$$

$$F_s^2 = V_s^{3,0} \oplus V_s^{2,1} = F_s^3 + \mathbb{C} \cdot \begin{pmatrix} 0 \\ 1 \\ a' \\ b' \end{pmatrix}$$

$$F_s^{\cancel{2}\,1} = F_s^2 + \mathbb{C} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ a'' \end{pmatrix}$$

where $a, a', \ldots$ are functions on D.

(1.1.4) $a, a', \dots$ holom. on S

(1.1.6.1)
$$(*)\begin{cases} a'' = -a \\ b' = b - aa' \end{cases}$$

(1.1.6.2) defines a nonempty open
$$\mathcal{U} \subset \{ (a, \dots) \text{ st. } (*) \}.$$

Note that $\dim_{\mathbb{C}} \mathcal{U} = 4$

(1.1.5) Write $\dot{a} = \frac{da}{ds}$, etc then we get
$$\begin{pmatrix} 0 \\ \dot{a} \\ \dot{b} \\ \dot{c} \end{pmatrix} \in F_s^2 , \quad \begin{pmatrix} 0 \\ 0 \\ \dot{a}' \\ \dot{b}' \end{pmatrix} \in F_s^1$$

This is equivalent to
$$\begin{cases} \dot{b} = \dot{a} a' \\ \dot{c} = \dot{a}(b - aa') \ [= \dot{a} b'] \end{cases}$$

Many solutions (even starting at the same point and having high order contact)

Prop 4.3.2 Suppose that

$$p : D^n \xrightarrow{a, a', \dots} U$$

corresponds to a PVHS over D^n.
Then $\dim p(D^n) \leq 1$.

Rem No "universal" PVHS's!

Proof. (1.1.5) says

$$\frac{\partial b}{\partial z_i} = a' \frac{\partial a}{\partial z_i}, \quad \frac{\partial c}{\partial z_i} = b' \frac{\partial a}{\partial z_i}.$$

Hence the vectors

$$\left(\frac{\partial a}{\partial z_i}, \frac{\partial b}{\partial z_i}, \frac{\partial c}{\partial z_i}\right) \in \mathbb{C} \cdot (1, a', b').$$

Hence

$$D^n \xrightarrow{a, b, c} \mathbb{C}^3$$

has image of dim ≤ 1. Finally

$$a' = \frac{\partial b}{\partial z_1} \Big/ \frac{\partial a}{\partial z_1}$$

depends on one parameter also

<u>Prop 4.3.3</u> The HS with a, b, c, a', b', a'' is **reducible** iff $\exists\ \lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{Q}$ such that

$$b = \lambda_1 + \lambda_2 a$$

$$c = \mu_1 + \mu_2 a$$

More precisely: $\dim_{\mathbb{Q}} \overbrace{\{1, a, b, c\}}^{(\mathbb{Q}\text{-span})} \leq 2.$

<u>Reference for CM question:</u> Ciprian Borcea, CY threefolds and CM.

<u>Reference for AH-cycles:</u> P. Deligne (notes by J. Milne), Hodge cycles on Abelian Varieties in LNM 900.