Outline of lectures on Model Theory and Diophantine Geometry, by Anand Pillay and Thomas Scanlon

1 Course description.

The general theme will be the use of model-theoretic methods in proofs of the Manin-Mumford conjecture and variants.
If K is an algebraically closed field, G a commutative algebraic group over K and Γ an abstract subgroup of $G(K)$, we will say that Γ is of *Lang type*, if for any n and subvariety X of G^n, $X(K) \cap \Gamma^n$ is a finite union of cosets of subgroups of Γ^n. The Manin-Mumford conjecture says that if the characteristic is 0 and G is a semiabelian variety, then the group $Tor(G)$ of torsion points of $G(K)$ is of Lang type.
Hrushovski [7] gave a proof of the Manin-Mumford conjecture, using the model theory of difference fields [3]. Scanlon [13] proved "Manin-Mumford for Drinfeld modules" in positive characteristic (see below), also using the model theory of difference fields [4].

Pillay's lectures will contain a simplified and self-contained account of Hrushovski's proof of Manin-Mumford in the spirit of [9]. A key step will be an elementary proof of the following algebraic result, valid in all characteristics:

Proposition 1.1 *Let A be a semiabelian variety and X be an irreducible subvariety of A containing 0 which generates A. Assume $Stab_A(X)$ is trivial. Let ϕ be a separable isogeny of A such that $\phi(X) = X$. Then for some n, ϕ^n is the identity.*

The methods, coming from [10], are closely related to those in recent preprints of Pink and Roessler ([11], [12]), but definability in difference fields plays a

simplifying role.
Pillay will also discuss how and why the statements of Manin-Mumford and Mordell-Lang, together with uniform definability of types in algebraically closed fields, automatically yield uniformities as the subvariety X of A varies in an algebraic family.

Scanlon's talks will be around the Drinfeld modules version of Manin-Mumford. Let K be a field of characteristic $p > 0$. The ring of endomorphism of the additive group $\mathbf{G}_a$ defined over K can be identified with the "twisted" polynomial ring $K\{\tau\} = \{a_0 + a_1\tau + .. + a_n\tau^n : n \in \mathbf{N}, a_i \in K\}$, where τ acts as $x \to x^p$. $\pi : K\{\tau\} \to K$ takes a polynomial f as above to its constant term a_0. By a *Drinfeld module* we will mean a homomorphism $\phi : \mathbf{F}_p[t] \to K\{\tau\}$ such that $\phi(t) \notin K$. The Drinfeld module ϕ equips the additive group of any K-algebra with an $\mathbf{F}_p[t]$-module structure. The Drinfeld module ϕ is said to have *generic characteristic* if the kernel of $\pi \circ \phi$ is (0).

Denis [5] raised a series of conjectures for Drinfeld modules, analogous to the Manin-Mumford, Mordell-Lang conjectures. Scanlon will explain the proof of the Drinfeld module version of Manin-Mumford:

Proposition 1.2 *Let K be an algebraically closed field of characteristic $p > 0$, and $\phi : \mathbf{F}_p[t] \to K\{\tau\}$ a Drinfeld module of generic characteristic. Then The $\mathbf{F}_p[t]$-torsion submodule of $(K, +)$ is of Lang type.*

Scanlon will also explain how the Drinfeld module version of Mordell-Lang with division points follows from the Drinfeld module version of Mordell-Lang without division points.

PREREQUISITES. Familiarity with the language of algebraic geometry, as in say Shafarevich [14], as well as with first order model theory. An acquaintance with the first five chapters of [1], by Bouscaren, Ziegler, Lascar, Pillay, and Hindry, respectively, would be helpful. A good reference on Drinfeld modules is Goss' book [6], specifically pages 63-92 in Chapter 4.

2 Project

Theorem 3.1 of [12] gives a generalization of Proposition 1.1 above to the case where ϕ is not necessarily separable. One case of this generalization is:

Proposition 2.1 *(char $= p > 0$.) Let A be a semiabelian variety, ϕ an isogeny of A, and X a ϕ-invariant subvariety of A containing 0. Assume that X generates A. Let $Frob$ the the Frobenius map, and suppose that for positive integers r, s, ϕ^s is the composition of $Frob^r$ with a separable isogeny from $Frob^r(A)$ to A. Then A can be defined over $\mathbf{F}_{p^r}$ and $\phi^s = Frob^r$ on A.*

Project 1 will to be generalize the proof of Proposition 1.1 in the lectures to a proof of Proposition 2.1

Project 2 will be to prove a "function field" version of Denis' Mordell-Lang conjecture. We call Drinfeld modules ϕ and ψ equivalent if there is a scalar λ such that $\psi(t) = \lambda^{-1}\phi(t)\lambda$. We define the modular transcendence degree of ϕ to be the least transcendence degree of a field L such that ϕ is equivalent to a Drinfeld module ψ defined over L. The project is to prove the following conjecture.

Conjecture 2.2 *Let K be an algebraically closed field of characteristic $p > 0$, ϕ a Drinfeld module over K of positive modular transcendence degree with $\pi \circ \phi(t) = 0$, and $\Gamma \leq (K, +)$ a finitely generated $\mathbf{F}_p[t]$-module. Then Γ is of Lang type.*

It should be possible to prove Conjecture 2.2 along the lines of Hrushovski's proof [8] of the function field Mordell-Lang. That is, working in a suitable separably closed field L of finite Ersov invariant over which the data are defined, let $\phi^\sharp(L)$ the type-definable group $\cap_n \phi(t)^n(L)$ and one has to prove the modularity of $\phi^\sharp(L)$. By [2], this is equivalent to the nonexistence of a definable isogeny between $\phi^\sharp(L)$ and L^{p^∞}.

References

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