Elementary equivalence versus Isomorphism

by Florian Pop at Bonn

Lecture 1: Introduction and motivation / The Question

It is one of the basic questions of Algebra to "classify" algebraic objects, like for instance fields, up to isomorphism. There are several aspects of this question, the main one being a *formalized definition of "classifying"*. Rather than riding on such themes, we will start by giving two typical examples:

1) The isomorphy type of a finite field K is given by its cardinality $|K|$, i.e., if K and L are such fields, then $K \cong L$ iff $|K| = |L|$.

1) The isomorphy type of an algebraically closed field K is determined by two invariants: (i) Absolute transcendence degree $\mathrm{td}(K)$, (ii) The characteristic $p = \mathrm{char}(K) \geq 0$. In other words, if K and L are algebraically closed fields, then $K \cong L$ iff $\mathrm{td}(K) = \mathrm{td}(L)$ and $\mathrm{char}(K) = \mathrm{char}(L)$.

Nevertheless, if we want to classify fields K up to isomorphism even in an "only a little bit" more general context, then we run into very serious difficulties... A typical example here is the attempt to give the isomorphy types of *real closed fields* (it seems first tried by Artin and Schreier). A real closed field K is "as close as possible" to its algebraic closure $\overline{K}$, as $[\overline{K} : K] = 2$, and $\overline{K} = K[\sqrt{-1}]$. These fields were introduced by Artin in his famous proof of Hilbert's 17^{th} Problem. Roughly speaking, the real closed fields are the fields having exactly the same *algebraic properties* as the reals $\mathbb{R}$. One knows quite a lot about real closed fields, see e.g. Prestel–Roquette [P–R], but for specialists it is clear that the problem of describing the isomorphism types of all such fields is hopeless... And to move into more modern times, the same is true for the p-adically closed fields (which roughly speaking, are the fields having exactly the same *algebraic properties* as the p-adics $\mathbb{Q}_p$, see loc.cit. above).

This is the place where the *model theory* enters the scene. The idea is that one should introduce a classification criterion of algebraic objects —in our case the fields— which is not as fine as the isomorphism, thus *coarser* than isomorphism, but which is still powerful enough to distinguish between "distinct field structures". This new criterion asks for classifying fields up to *elementary equivalence,* a term which we explain below.

First, let $\mathcal{L}_{\mathrm{fields}}$ be the language of fields. Thus this is the language in which we express the *axioms* the composition laws Addition $+$ and Multiplication $\cdot$ have to satisfy in order to have a "field structure". Let further $\mathcal{A}_{\mathrm{fields}}$ be the set of all the sentences, i.e., parameter free formulas, in the language of fields. Finally, for every field K, we denote by $\mathfrak{Th}(K)$ the set of all sentences $\varphi \in \mathcal{A}_{\text{fields}}$ which are true in K; symbolically written:

$$\mathfrak{Th}(K) = \{\, \varphi \in \mathcal{A}_{\text{fields}} \mid K \models \varphi \,\}$$

We call $\mathfrak{Th}(K)$ the *elementary theory* of K. Further, we say that two fields K and L are *elementarily equivalent,* if $\mathfrak{Th}(K) = \mathfrak{Th}(L)$, i.e., if they have the same elementary theory; or in other words, if exactly the same parameter free formulas (in the language of fields) are true in both K and L.

Remarks:

Suppose that K and L are isomorphic fields. Then $\mathfrak{Th}(K) = \mathfrak{Th}(L)$, i.e., isomorphic fields have the same elementary theory and are elementarily equivalent. In particular, classifying fields up elementary equivalence is *coarser,* (thus easier) than classifying fields up to isomorphism. Here is a short list of what can "see" the elementary theory of a field:

1) The characteristic of a field K is obviously encoded in $\mathfrak{Th}(K)$, i.e., if $\mathfrak{Th}(K) = \mathfrak{Th}(L)$, then $\text{char}(K) = \text{char}(L)$ (WHY?).

2) Being algebraically closed is obviously encoded in $\mathfrak{Th}(K)$, i.e., if K is algebraically closed, and $\mathfrak{Th}(K) = \mathfrak{Th}(L)$, then L is algebraically closed (WHY?).

3) Being real closed is encoded in $\mathfrak{Th}(K)$, i.e., if $\mathfrak{Th}(K) = \mathfrak{Th}(L)$, and if K is real closed, then L is real closed (WHY?). [**Hint:** Recall the "characterization" of the real closed fields: K is real closed iff (i) Every sum of squares is again a square, (ii) Every polynomial of odd degree has a root in K, (iii) -1 is not a square in K. Clearly, the fact that the "axioms" (i), (ii), (iii), are true or not is encoded in $\mathfrak{Th}(K)$.]

4) Being p-adically closed is encoded in $\mathfrak{Th}(K)$, i.e., if K is p-adically closed, and $\mathfrak{Th}(K) = \mathfrak{Th}(L)$, then L is p-adically closed (WHY?). [**Hint:** This follows from the characterization of p-adically closed fields, see for instance [P–R]. We should remark that the characterization of p-adically closed fields is similar but though a little bit more complicated than the one of the real closed fields. It uses the so called *MacIntyre's predicates* P_n $(n \geq 1)$, etc..]

5) A *geometric interpretation* of the elementary equivalence is the following: Let $\varphi \in \mathcal{A}_{\text{fields}}$ be a given sentence, say with variables $X_1, \dots, X_n$. Then φ describes a "constructible subset" $\mathcal{S}_\varphi \subset \mathbb{A}^n$ defined over $\mathbb{Z}$. Now one has:

Theorem. *Two fields K and L are elementarily equivalent if and only if for all $\varphi \in \mathcal{A}$ it holds: $\mathcal{S}_\varphi(K) \neq \emptyset \Leftrightarrow \mathcal{S}_\varphi(L) \neq \emptyset$.*

6) Finally, the precise relation between elementary equivalence and isomorphism is given via *ultra-powers* as follows. First, we recall the notion of an ultrapower of a field: Let K be a given field, I an index set, and $\mathcal{D}$ an ultrafilter on I.

We define the *ultra-power* $K^* = K^I/\mathcal{D}$ of K with respect to $\mathcal{D}$ to be $K^I/\sim$, where $\sim$ is the equivalence relation on K^I defined by: $(x_i)_i \sim (y_i)_i \Leftrightarrow \{\, i \,|\, x_i = y_i \,\} \in \mathcal{D}$. It turns out that K^* carries in a canonical way a field structure; and the canonical projection $K^I \to K^*$ is a morphism of K-algebras, if we endow K^I with the diagonal embedding $K \hookrightarrow K^I$. We will call the resulting embedding $K \hookrightarrow K^*$ the canonical *diagonal embedding.* In algebraic terms, one can describe the projection $K^I \to K^*$ as follows: Its kernel $\mathfrak{m}_{\mathcal{D}}$ is the set of all the elements $(x_i)_i$ whose support does not lie in $\mathcal{D}$. And conversely, any prime ideal $\mathfrak{n}$ of R^I is a maximal ideal, and the supports of all the elements $(x_i)_i$ which do not belong to $\mathfrak{n}$ build an ultrafilter $\mathcal{D}_{\mathfrak{n}}$ on I, such that $\mathfrak{m}_{\mathcal{D}_{\mathfrak{n}}} = \mathfrak{n}$. We will say that the ultra-power K^* is trivial, if the ultra-filter $\mathcal{D}$ is trivial, i.e., $\exists\, i \in I$ such that $\mathcal{D}$ consists of all the subsets $I' \subset I$ with $i \in I'$. Equivalently, $\mathcal{D}$ is trivial as above if and only if $\mathfrak{m}_{\mathcal{D}}$ is the kernel of the i^{th} structural projection $K^I \to K$ for some $i \in I$. Thus if this is the case, then $K^* \cong K$ as a K-algebra, thus canonically.

We remark that directly by the definition of $\mathfrak{Th}(K)$ and of the ultra-power K^* one has: $\mathfrak{Th}(K) = \mathfrak{Th}(K^*)$, thus K and K^* are elementarily equivalent. Further, the precise relation between elementary equivalence and isomorphism is the following:

Theorem. *Two fields K and L are elementarily equivalent iff there exist ultra-powers $K^* = K^I/\mathcal{D}$ and $L^* = L^J/\mathcal{E}$ which are isomorphic.*

In particular, by the above Theorem, two elementarily equivalent fields K and L have isomorphic *absolute subfields* $K^{\text{abs}} \cong L^{\text{abs}}$. Here, the absolute field K^{abs} of a field K is by definition the relative algebraic closure of the prime field in K. For instance, $\mathbb{R}^{\text{abs}}$ consists of all the algebraic real numbers in $\mathbb{R}$, whereas $\mathbb{Q}_p^{\text{abs}}$ consists of all the algebraic p-adic numbers in $\mathbb{Q}_p$, and finally, $\mathbb{C}^{\text{abs}}$ is an algebraic closure of $\mathbb{Q}$.

I think it is interesting to remark that using ultra-powers of fields, some famous conjectures/results have a "nice" interpretation as follows: Recall the *Mordell Conjecture* (now a Theorem of Faltings), which asserts that every irreducible curve $X \to K$ of genus $g > 1$ over a number field K has only finitely many K-rational points. This is equivalent to the following fact: *The function field $\kappa(X)$ of X cannot be embedded in any ultra-power K^*.*[1]

7) It is one of the basic facts of the model theoretic algebra to show the following:

[1] The so called "non-standard method" developed by Robinson, Robinson–Roquette, Kani, etc., —which would provide a different approach to prove the Mordell Conjecture— aims actually to showing that this last assertion is true...

Theorem.

(1) *Two algebraically closed fields K and L are elementarily equivalent iff they have the same characteristic. Thus $\overline{\mathbb{Q}}$ and $\overline{\mathbb{F}}_p$ (all primes p) are representatives for the elementary equivalence classes of all algebraically closed fields.*

(2) *All real closed fields are elementarily equivalent. The field $\mathbb{R}^{\rm abs}$ of all real algebraic numbers is real closed and contained in all real closed fields. Finally $\mathbb{R}^{\rm abs}$ is a representative for the elementary equivalence class of all real closed fields.*

(3) *All p-adically closed fields are elementarily equivalent. The field $\mathbb{Q}_p^{\rm abs}$ of all real p-adic numbers is p-adically closed and contained in all p-adic closed fields. Finally $\mathbb{Q}_p^{\rm abs}$ is a representative for the elementary equivalence class of all p-adically closed fields (each p).*

A way to view the Theorem above is that —in contrast to the classification up the isomorphism—, the "sufficiently localized" fields,[2] e.g. the *algebraically closed,* and/or the *real* and/or *p-adically closed* ones have a very easy classification up the elementary equivalence: The elementary equivalence classes are exactly the elementary equivalence classes of the absolute parts of these fields: The algebraic closures $\overline{\mathbb{Q}}$, respectively $\overline{\mathbb{F}}_p$ (all p), and respectively $\mathbb{R}^{\rm abs}$ and $\mathbb{Q}_p^{\rm abs}$.

Therefore, the natural question arises about the nature of the elementary equivalence classes of the "most global" fields, which are the arithmetic function fields, and in a precise sense also the geometric function fields.

First let us recall the following general facts about function fields:

Let k be a given base field. A function field $K|k$ is a field extension with K a finitely generated field over k. A model of $K|k$ is any irreducible variety $X \to k$ such that $k(X) = K$. We denote by $d = \mathrm{td}(K|k) = \dim(X)$ the transcendence degree of $K|k$. Any two models $X \to k$ and $Y \to k$ of $K|k$ are birationally equivalent over k, i.e., there exist dominant rational maps $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow X$ such that $f \circ g$ and $g \circ f$ are equivalent as rational maps to id_Y, respectively id_X. Finally, the category of all function fields over k and k-embeddings of fields is equivalent to the category of irreducible k-varieties and dominant rational k-maps.

Finally, given two function fields $K|k$ and $L|l$, an embedding of function fields $\imath : K|k \hookrightarrow L|l$ is a field embedding mapping k into l such that the compositum $K_\imath = \imath(K) \cdot l$ is a function field over l with $\mathrm{td}(K_\imath|l) = \mathrm{td}(K|k)$. Equivalently, if $(t_1, \dots, t_d)$ is a transcendence basis of $K|k$, then its image $\imath(t_1, \dots, t_r)$ is contained in a transcendence basis of $L|l$. In geometric terms, giving an embedding of function fields $\imath : K|k \hookrightarrow L|l$ is equivalent to giving the following: First, models $X \to k$ and $Y \to l$ for $K|k$ respectively $L|l$; second, an embedding of fields

[2] The precise sense of "localizing" here comes from the étale topology —at least in the characteristic zero case—...

$\imath_0 : k \to l$; and third, a dominant rational map $Y \dashrightarrow X_\imath$, where $X_\imath$ is one of the irreducible components of the base change $X_l := X \times_{\imath_0} l$ of X to l via $\imath_0$.

Definition.

1) An *arithmetic function field* over a prime field k is a finitely generated field extension $K|k$. Equivalently, K is the function field of an irreducible k-variety $X \to k$.

2) A *geometric function field* over an algebraically closed field k is a finitely generated field extension $K|k$. Equivalently, K is the function field of an irreducible k-variety $X \to k$.

We denote by $\mathcal{F}$ the class of all function fields, and by $\mathcal{F}_{\text{arith}}$ and $\mathcal{F}_{\text{geom}}$ the sub-classes of all arithmetic, respectively geometric, function fields.

Problem: Elementary equivalence versus Isomorphisms (EeVIP)

Let K be a given arithmetic or geometric function field. Describe the elementary equivalence class of K inside $\mathcal{F}$.

Conjecturally, this class consists of K itself. In other words, if L is another such field, then $\mathfrak{Th}(K) = \mathfrak{Th}(L)$ if and only if $K \cong L$ as fields.

I learned the above question in the arithmetic case from Sabbagh in the Eighties. And it seems that the first who tried to tackle this Problem in the geometric case was Duret [D1], [D2], who worked under *the hypothesis* $\mathrm{td}(K|k) = 1$. Thus $K|k$ is the function field of a projective smooth curve $X \to k$, say of genus g. Duret showed that the EeVIP has a positive answer in the case $g \neq 1$, and further, in the case $\mathrm{char}(k) = 0$, if X is an elliptic curve without complex multiplication. This was extended by Pierce [Pi] to all characteristics.

To start with, let us mention the first major difficulty one has to overcome when trying to tackle the above problem in general: It is to show that the transcendence degree $d = \mathrm{td}(K|k)$ of the function field in discussion is encoded in the elementary theory $\mathfrak{Th}(K)$ of K, and more precisely, to saying that a given system of elements $(t_1, \dots, t_d)$ is a separable transcendence basis of the function field in discussion.[3]

Now the main results we want to explain in these lectures are roughly speaking the following:

Theorem A (Arithmetic variant). *Let K and L be arithmetic function fields which are elementarily equivalent. Let k and l be their absolute subfields. Then one has:*

[3] There is a fine difference between the two questions: The fact that the transcendence degree is d might be expressible by an infinite "scheme of axioms ", like for instance, the fact that a field has characteristic zero. In contrast to this, to say that $(t_1, \dots, t_d)$ is a transcendence basis of $K|k$ must be expressed by a single formula in which $t_1, \dots, t_d$ are the only parameters...

(1) *k and l are isomorphic, and* $\mathrm{td}(K|k)$ *equals* $\mathrm{td}(L|l)$.

(2) *Moreover, there exists a field embedding $\imath : K \to L$ such that L is finite separable over $\imath(K)$.*

In particular, if K is of general type, then $K \cong L$ as fields.

Theorem B (Geometric variant). *Let $K|k$ and $L|l$ be function fields over algebraically closed fields k, respectively l. Suppose that K and L are elementarily equivalent as fields. Then one has:*

(1) *k and l are elementarily equivalent, and* $\mathrm{td}(K|k)$ *equals* $\mathrm{td}(L|l)$.

(2) *Suppose that $K|k$ is of general type. Then there exist function subfields $K_0|k_0 \hookrightarrow K|k$ and $L_0|l_0 \hookrightarrow L|l$ such that $K = K_0\, k$ and $L = L_0\, l$, and further $K_0|k_0 \cong L_0|l_0$ as function fields.*

In particular, if $k \cong l$ are isomorphic, then $K|k \cong L|l$ are isomorphic as function fields.

In the above Theorems, we call a function field $K|k$ of general type, if it has projective smooth models $X \to k$, which are varieties of general type. For instance, a projective smooth curve $X \to k$ is of general type if and only if its genus satisfies $g > 1$. Thus the results of Duret and Pierce mentioned above go beyond Theorem B above in the case of $\mathrm{td}(K|k) = 1$.

Nevertheless, it remains an **open question** to give the precise *relation between elementary equivalence and isomorphism* in the "non-general" case.

We would also like to mention that a preliminary form of the above results, more precisely, Theorem A in the case $\mathrm{td}(K|k) \leq 1$, were known to the author already in the Nineties, but the approach was different, see [P2]. The main tool was the Mordell Conjecture (as proved by Faltings). In particular, we exploited the relation between *rational points* on general curves and the *elementary theory* of function fields of curves over number fields. It remains a serious **open question** to understand in general the relation between these two apparently different questions. Finally, B. Poonen communicated to me (private, Sept. 2001) that he also has a proof of the Theorem A above (relaying on arguments from [P2]).

Lecture 2: Detecting td( ) and transcendence bases

In this Section we explain how one can describe by a relatively simple assertion in the language of fields the fact that an arithmetic/geometric function field $K|k$ has a transcendence basis $\mathcal{B} = (t_1, \dots, t_d)$. In particular, from this it follows that the fact $\mathrm{td}(K|k) = d$ is expressible by a relatively simple sentence in the language of fields.

We should remark that in the positive characteristic case, both for arithmetic and geometric function fields, as well as for geometric function fields, the answer to this question about detecting transcendence bases is "easy". Nevertheless, for function fields over $\mathbb{Q}$, the only way we can do this is by using the *Milnor Conjecture* (as proved by Voevodsky, Rost, Voevodsky–Orlov–Vishik, et al, see Kahn's [Kh] talk in the Séminaire Bourbaki, and Pfister [Pf2]).

The idea is to use *quadratic Pfister forms* in characteristic zero, and a straightforward *generalization of these* in characteristic $p > 0$. And to ask that such forms have properties like being *universal,* and *do/do not represent* 0. But let first recall the definitions.

Definition. Let $q = q(X_1, \dots, X_n)$ be a homogeneous form over some base field K. Let $K'|K$ be a field extension. We will say that:

1) q is *universal over* K'*,* if the following $\forall\,\exists$*-formula over* K' is true in K': $\forall\, a \in K'^{\times}\ \exists\, a_1, \dots, a_n \in K'$ such that $q(a_1, \dots, a_n) = a$.

2) q *represents* 0 *over* K'*,* if the following $\exists$*-formula over* K' is true in K': $\exists\, a_1, \dots, a_n \in K'$, not all equal to 0, such that $q(a_1, \dots, a_n) = 0$.

Fact 1. *In particular, in the above context, let* $K' = K[\alpha]$*, with* α *some fixed absolute algebraic integer, e.g.,* $\alpha^2 = 1$*. Then the following holds:*

1) "q is universal over K'" *is a* $\forall\exists$*-formula in the language of fields having the coefficients of* q *as the only parameters.*

2) "q does not represent 0 over K'" *is a* $\forall$ *-formula in the language of fields having the coefficients of* q *as the only parameters.*

We will use this observation in the following context: Let $K|k$ be a function field with $\mathrm{td}(K|k) = d$. For a given system $(t_1, \dots, t_r) = \underline{t}$ of elements of $K^{\times}$, and a natural number p, we consider the following form of degree p in $n = p^r$ variables $X_{\underline{i}}$ over K:

$$q^{(p)}_{(t_1,\dots,t_r)} = \sum_{\underline{i}} \underline{t}^{\underline{i}} X^p_{\underline{i}}$$

where $\underline{i} = (i_1, \dots, i_r)$ $(0 \le i_1, \dots, i_r < p)$ and $\underline{t}^{\underline{i}} = t_1^{i_1} \dots t_d^{i_r}$. We remark that in the case $p = 2$, the resulting quadratic form $q_{(t_1,\dots,t_r)} := q^{(2)}_{(t_1,\dots,t_r)}$ defined above is the r*-fold Pfister form* $<< t_1, \dots, t_r >>$ attached to the system of elements $(t_1, \dots, t_r)$ of $K^{\times}$.

Taking into account that the forms $q^{(p)}_{(t_1,\dots,t_r)}$ have only the monomials $\underline{t}^{\underline{i}}$ as coefficients, by the Fact above we have the following:

Fact 2. *In the above context, let* p *be a fixed prime number, and* r *a fixed natural number. Let* $K' = K[\alpha]$*, with* α *some fixed absolute algebraic integer, e.g.,* $\alpha^2 = 1$*. Then the following holds:*

1) "$\forall\, t_1, \dots, t_d$, the resulting $q^{(p)}_{(t_1,\dots,t_r)}$ is universal over K'" *is a* $\forall\exists$*-sentence in the language of fields.*

2) "$\exists\, t_1, \dots, t_d$, such that the resulting $q^{(p)}_{(t_1,\dots,t_r)}$ is universal over K'" *is an* $\exists\,\forall\,\exists$ *-sentence in the language of fields.*

3) "$\exists\ t_1, \dots, t_d$ such that $q^{(p)}_{(t_1,\dots,t_r)}$ does not represent 0 over K'" *is an* $\exists\,\forall$ *-sentence in the language of fields.*

Case 1) *k has positive characteristic*

In the above context, suppose k is a perfect field of characteristic $p > 0$ (in particular, k might be finite or algebraically closed). We recall the following two definitions:

1) A system of elements $(t_1, \dots, t_r)$ of $K^\times$ is called *p-independent,* if the system of all the monomials $\underline{t}^{\underline{i}} = t_1^{i_1} \dots t_d^{i_r}$ $(0 \le i_1, \dots, i_r < p)$ is linearly independent over K^p.

2) A system of elements $(t_1, \dots, t_d)$ of $K^\times$ is said to be a *p-basis* of $K|k$, if the system of all the monomials $\underline{t}^{\underline{i}}$ is a vector space basis of K over K^p.

Both assertion 1) and assertion 2) can be interpreted as formulas in the language of fields with no parameters excepting $t_1, \dots, t_r$ by using the form $q^{(p)}_{(t_1,\dots,t_r)}$ introduced above. Namely for a given system $(t_1, \dots, t_r)$ of elements of $K^\times$ as above, one has: The form $q^{(p)}_{(t_1,\dots,t_r)}$ does not represent 0 over K if and only if $(t_1, \dots, t_r)$ is p-independent; and the form $q^{(p)}_{(t_1,\dots,t_d)}$ is universal over K, but does not represent 0 over K if and only if $(t_1, \dots, t_d)$ is a p-basis of $K|k$.

Finally, recall the relation of p-independence with the transcendence bases of a function field $K|k$, where k is a perfect field, like for instance a finite or an algebraically closed field, see e.g. [Ei], Appendix 1. For a system of elements $(t_1, \dots, t_r)$ of $K^\times$ the following hold:

1)′ $(t_1, \dots, t_r)$ is p-independent if and only if $(t_1, \dots, t_r)$ can be completed to a separable transcendence basis of $K|k$ if and only if $q^{(p)}_{(t_1,\dots,t_r)}$ does not represent 0 over K.

2)′ $(t_1, \dots, t_d)$ is a p-basis if and only if $(t_1, \dots, t_d)$ is a separable transcendence basis of $K|k$ iff the form $q^{(p)}_{(t_1,\dots,t_d)}$ is universal over K, but does not represent 0 over K.

Thus combining the facts above we get the following description of the separable transcendence bases of $K|k$:

Fact 3. *Let $K|k$ be a function field with k a perfect field of characteristic $p > 0$. The following hold:*

(1) *$K|k$ has transcendence degree d if and only if the following $\exists\forall\exists$-sentence is true in K:*

"$\exists\, t_1, \dots, t_d$ such that $q^{(p)}_{(t_1,\dots,t_d)}$ is universal over K, but does not represent 0 over K".

(2) *A system* $(t_1, \dots, t_d)$ *of elements of* $K^\times$ *is a separable transcendence basis of* $K|k$ *if and only if the sentence from* (1) *is true for* $(t_1, \dots, t_d)$.

Before analyzing the arithmetic case over $\mathbb{Q}$, respectively the geometric case for char $\neq 2$, let us mention the Lemma below for later use. First, we remark that the forms $q^{(p)}_{(t_1,\dots,t_r)}$ above are special cases of the so called "pure forms". The pure forms over an arbitrary field K are homogeneous forms of the shape $q = q(X_1, \dots, X_n) = a_1X_1^p + \dots + a_nX_n^p$. Thus q is defined by its coefficients $(a_1, \dots, a_n)$. The pure forms can be "added" and "multiplied" in an obvious way: Let namely q' be another such form of degree p, say with coefficients $(a'_1, \dots, a'_{n'})$. Then define $q \oplus q'$ to be the pure form of degree p in $(n+n')$ variables with coefficients $(a_1, \dots, a_n, a'_1, \dots, a'_{n'})$; and define $q \otimes q'$ to be the pure form of degree p in $n\,n'$ variables with coefficients $(a_1a'_1, \dots, a_na'_{n'})$. Clearly, $\oplus$ and $\otimes$ are associative and commutative, and $\otimes$ is distributive with respect to $\oplus$.

Notation/Remark: Let q be a given pure form of degree p with coefficients $(a_1, \dots, a_n)$, and $a \in K^\times$ given. Define: $<\!<a>\!>^{(p)} := q^{(p)}_{(a)}$ to be the pure form of degree p with coefficients $(1, a, \dots, a^{p-1})$.

In particular, for $m = 2$, the resulting $<\!<a>\!> := <\!<a>\!>^{(2)} = X_1^2 + aX_2^2$ is the usual 1-fold Pfister form attached to (a).

And in general, $q^{(p)}_{(t_1,\dots,t_r)} = <\!<t_1>\!>^{(p)} \otimes \dots \otimes <\!<t_r>\!>^{(p)}$.

Lemma 4. *Let* K *be a discrete valued field with residue field* K_0 *and uniformizing parameter* π. *Let* q_0 *be a pure form of degree* p *over* K_0 *which does not represent* 0 *over* K_0. *For some lifting* $\tilde{q}_0$ *of* q_0 *to* K, *let us denote* $q = <\!<\pi>\!>^{(p)} \otimes \tilde{q}_0$. *Then if* q *does not represent* 0 *over* K.

Proof. Exercise.

Case 2) *k is algebraically closed,* $\mathrm{char}(k) \neq 2$

Suppose that $K|k$ is a function field with k an algebraically closed field of characteristic $\neq 2$. Using the Pfister forms $q_{(t_1,\dots,t_r)} := q^{(2)}_{(t_1,\dots,t_r)}$ defined above, we can characterize the transcendence degree of $K|k$, and even more, determine whether a system of elements $(t_1, \dots, t_d)$ of $K^\times$ is a transcendence basis of $K|k$ as follows.

The role of the theory of p-bases from the characteristic $p > 0$ case above is played now by another elementary property of fields as follows. Let us first recall that a field K is called a *C_r-field,* the following holds: *Every form* $q(X_1, \dots, X_n)$ *of degree* p *represents* 0, *provided* $n > p^r$. The following facts are well known, see e.g. LANG [L1], or [S1], II, § 3–4:

1) Some prominent examples:

- The finite fields are C_1-fields (CHEVALLEY–WARNING); see e.g. [S3].

- The complete discrete valued fields with algebraically closed residue field are C_1-fields (LANG's Thesis).

• It is a famous open question, the *Artin's Conjecture,* whether $\mathbb{Q}^{\rm ab}$ is a C_1-field.

2) Let K be a C_r-field. Then every algebraic extension $K'|K$ of K is a C_r-field too (LANG's Thesis).

3) Let k be a C_r-field, and let $K|k$ be a field extension with $\mathrm{td}(K|k) = d$. Then K is a C_{r+d} field (LANG's Thesis, + NAGATA).

In particular, let k be an algebraically closed field, and $K|k$ be an arbitrary field extension with $\mathrm{td}(K|k) = d$. Then K is a C_d-field

Fact 5. *Let $K|k$ be a function field with k an algebraically closed field with* $\mathrm{char}(k) \neq 2$*. In notations as above the following hold:*

(1) *$K|k$ has transcendence degree d and only if the following conjunction of a $\forall\,\exists$ -sentence and an $\exists\,\forall$ -sentences in the language of fields is true in K:*

" $\forall\, t_1, \dots, t_d$ the Pfister from $q_{(t_1,\dots,t_d)}$ is universal over K and $\exists\, t_1, \dots, t_d$ such that $q_{(t_1,\dots,t_d)}$ does not represent 0 over K ".

(2) *Suppose that* $\mathrm{td}(K|k) = d$*. Let* $(t_1, \dots, t_d)$ *be a system of elements of* $K^\times$ *as above such that* $q_{(t_1,\dots,t_d)}$ *does not represent* 0 *over* K *. Then* $(t_1, \dots, t_d)$ *is a transcendence basis of* $K|k$*.*

Zusatz: *Let* $(t_1, \dots, t_d)$ *be a separable transcendence basis of* $K|k$*. Then "for almost all" tuples* $(a_1, \dots, a_d)$ *with* $a_i \in k$*, the resulting d-fold Pfister form* $q_{(t_1-a_1,\dots,t_d-a_d)}$ *does not represent* 0 *over* K*.*

Proof. (See [P1], Section 1, Fact 1.2.) Let $d = \mathrm{td}(K|k)$. We first show that $q_{(t_1,\dots,t_d)}$ is universal over K for every $(t_1, \dots, t_d)$. Namely, for every $a \in K^\times$, consider the following quadratic form $q = aX_0^2 - q_{(t_1,\dots,t_d)}$. Then q has $2^d + 1$ variables, hence it represents 0 over K. Since $\mathrm{char}(k) \neq 2$, from this it follows that $q_{(t_1,\dots,t_d)}$ represents a over K (WHY?).

We next show that for a properly chosen $(t_1, \dots, t_d)$, the resulting Pfister form $q_{(t_1,\dots,t_d)}$ does not represent 0 over K. In order to do this, let $(t_1, \dots, t_d)$ be an arbitrary separable transcendence basis for $K|k$. Set $R = k[t_1, \dots, t_d]$, and denote by S the integral closure of R in K. Then the resulting homomorphism of k-algebras $R \hookrightarrow S$ is étale on a Zariski open subset of $\mathrm{Spec}\, R \cong \mathbb{A}^d_k$. Thus for "almost all" $\underline{a} \in \mathbb{A}^d_k(k)$, the resulting system of elements $(t_1 - a_1, \dots, t_d - a_d)$ is a system of parameters at every point of $\mathrm{Spec}\, S$ which lies over $\underline{a}$. Let $x \in \mathrm{Spec}\, S$ be such a closed point, and $(\mathcal{O}_x, \mathfrak{m}_x)$ its local ring. Then the $\mathfrak{m}_x$-adic completion of $(\mathcal{O}_x, \mathfrak{m}_x)$ is k-isomorphic to $k[[t_1 - a_1, \dots, t_d - a_d]]$. From this we deduce that K is k-embeddable into $\Lambda := k((t_1 - a_1)) \dots ((t_d - a_d))$, the field of iterated Laurent power series in $(t_1 - a_1, \dots, t_d - a_d)$. Now in order to conclude that $q_{(t_1,\dots,t_d)}$ does not represent 0 over K, it is sufficient to show that $q_{(t_1,\dots,t_d)}$ does not represent 0 over Λ. This last assertion follows by induction from the Lemma 4 above.

In order to finish the proof of Fact 5, we have to show that no natural number $r < \mathrm{td}(K|k)$ has the property that for every system $(t_1, \dots, t_r)$ of elements of $K^\times$, the resulting $q_{(t_1,\dots,t_r)}$ is universal. This is an immediate consequence of the discussion above: Let $(t_1, \dots, t_d)$ be a transcendence basis such that $q_{(t_1,\dots,t_r)}$ does not represent 0 over K. Let $(t_1, \dots, t_r)$ be its sub-system with $r < d$. Then by the general theory of Pfister forms we have: Since $q_{(t_1,\dots,t_d)}$ does not represent 0 over K, the sub-form $q_{(t_1,\dots,t_r)}$ does not represent t_d over K.

Lecture 3: Continuation... / Proof of Thm A

Case 3) *k is a number field*

We now come to discussing the arithmetic case, which is the most interesting (but most difficult) one. Thus here $K|k$ is a function field over a number field k. The proof is quite technical, and relies on deep results relating the transcendence degree to properties of Pfister forms via a *cohomological characterization* of td( ) and the *Milnor Conjecture*... Therefore we will only sketch the proof here, see [P1] for details.

a) *Cohomological characterization of* $\mathrm{td}(K|k)$

For a function field $K|k$ as above, and a prime number ℓ, we denote by $\mathrm{vcd}_\ell(K)$ the virtual ℓ-cohomological dimension of K. It actually equals $\mathrm{cd}_\ell(E)$ for any non-real finite extension $E|K$. In particular, if $\sqrt{-1}$ lies in the field in discussion, then $\mathrm{vcd}_\ell = \mathrm{cd}_\ell$ for every $\ell \neq \mathrm{char}$. By classical results in Galois cohomology, see e.g. Serre [S1], Ch.II, one has:

- If $\ell = \mathrm{char}(K)$, then $\mathrm{cd}_\ell(K) = 1$ (this is a theorem of Shafarevich).

- If K is finite, then $\mathrm{vcd}_\ell(K) = 1$ for all ℓ. And if K is a number field, then $\mathrm{vcd}_\ell(K) = 2$ for all ℓ (this is a theorem of Tate).

Combined with the fact that $\mathrm{vcd}_\ell(K) = \mathrm{vcd}_\ell(k) + \mathrm{td}(K|k)$ for all function fields $K|k$, and all $\ell \neq \mathrm{char}(k)$, we finally get:

Fact 6. *Let $K|k$ be a function field over a number field k. Then one has:* $\mathrm{td}(K|k) = d = \mathrm{cd}_\ell(E) - 2$ *for every finite field extension $E|K$ such that $\sqrt{-1} \in E$.*

We next want to give a criterion to detect the transcendence degree d above in pure cohomological terms. In order to do this, let μ_ℓ be the group of ℓ^{th} roots of unity. Via the canonical Kummer theory isomorphism $\delta : E^\times/\ell \to \mathrm{H}^1(G_E, \mu_\ell)$, we get canonical homomorphisms

$$(E^\times)^n \to (E^\times/\,\ell\,)^n \to \mathrm{H}^1(G_E,\mu_\ell)^{\,n} \xrightarrow{\cup} \mathrm{H}^n(G_E, \mu_\ell^{\otimes n}),$$

defined/denoted by $(a_1, \dots, a_n) \mapsto a_1 \cup \dots \cup a_n$. We further remark, that if $\mu_\ell \subset E$, then $\mathrm{H}^1(G_E, \mu_\ell) = \mathrm{Hom}(G_E, \mathbb{Z}/\ell)$ is the character group of G_E with values in

$\mathbb{Z}/\ell$, and second, $\mathrm{H}^n(G_E, \mu_\ell^{\otimes n})$ can be identified with $\mathrm{H}^n(G_E, \mathbb{Z}/\ell)$. The main observation is now the following

Lemma 7. *In the context of Fact above, let ℓ be a prime number. Let E be a finite extension of K containing $\mu_{2\ell}$. Then $m = \mathrm{vcd}_\ell(K)$ is the unique natural number m with the following properties:*

(i) *There exist $a_1, \dots, a_m$ in $K^\times$ such that $a_1 \cup \ldots \cup a_m \neq 0$ as an element of $\mathrm{H}^m(G_E, \mathbb{Z}/\ell)$.*

(ii) *For arbitrary elements $a_1 \dots, a_{m+1} \in K^\times$, one has: The cup product $a_1 \cup \ldots \cup a_{m+1} = 0$ in $\mathrm{H}^{m+1}(G_E, \mathbb{Z}/\ell)$.*

Proof. See [P1], Section 1, C), Proof of Fact 1.3.

b) *The Milnor Conjecture and* $\mathrm{td}(K|k)$

Let $\mathrm{K}_n^{\mathrm{M}}(E)$ denote the n^{tn} Milnor K-group of a field E. Recall that the canonical isomorphism $\mathrm{K}_1^{\mathrm{M}}(E)/\ell \cong E^\times/\ell \cong \mathrm{H}^1(G_E, \mu_\ell)$ gives rise to the tame symbol from the Milnor K-theory, which is a canonical homomorphisms

$$\mathrm{K}_n^{\mathrm{M}}(E)/\ell \xrightarrow{h} \mathrm{H}^n(G_E, \mu_\ell^{\otimes n}),$$

defined by $\{a_1, \dots, a_n\} \mapsto a_1 \cup \ldots \cup a_n$, see e.g. Milnor [M1]. It is conjectured that h is an isomorphism for all ℓ prime to $\mathrm{char}(E)$. This is a generalization of the so called *Milnor Conjecture,* which is the above assertion for $\ell = 2$. The point is that the Milnor Conjecture has a deep arithmetic significance related to the arithmetic of the quadratic forms (for general ℓ we do not have yet an interpretation of the "generalized Milnor Conjecture"). The Milnor conjecture is now proved by contributions of several people, with the last major steps by Voevodsky [V1], Rost [R1], [R2], Voevodsky–Orlov–Vishik [OVV], et al, see Kahn [Kh]. We describe below the facts which are significant for us.

With E as above, let $W(E)$ be the Witt ring of E, i.e., the set of the isomorphy classes of anisotropic quadratic forms over E with the usual addition and multiplication. Let $I(E)$ be the ideal of even dimensional quadratic forms, and $I^n(E)$ its powers. The set of all$n$-fold Pfister forms generates $I^n(E)/I^{n+1}(E)$. Milnor defined for every n a homomorphism $d_n : \mathrm{K}_n^{\mathrm{M}}(E)/2 \to I^n(E)/I^{n+1}(E)$ and conjectured that both homomorphisms $h_n : \mathrm{K}_n^{\mathrm{M}}(E)/2 \to \mathrm{H}^n(G_E, \mathbb{Z}/2)$, and d_n are in fact isomorphisms. In particular, this would give rise to *higher cohomological invariants* for quadratic forms $e_n : I^n(E)/I^{n+1}(E) \to \mathrm{H}^n(G_E, \mathbb{Z}/2)$ for every n, thus generalizing the dim(mod 2), the discriminant, and the Clifford (Hasse-Witt) invariant. The above isomorphism would work at the level of the Pfister forms as follows, see Elman–Lam [E–L], Jacobs–Rost, [J–R], etc. for more details and literature:

$$e_n : \ll a_1, \dots, a_n \gg \ \mapsto (-a_1) \cup \ldots \cup (-a_n).$$

(Note that the minus-sign comes from a convention which is not necessarily the same in all sources. It depends on the definition of $\ll a \gg$ which is either

$\ll a \gg \; = \; X_0^2 + aX_1^2$ or $\ll a \gg \; = \; X_0^2 + aX_1^2$. We work with the "+" convention.) We recall the following fact: Let $q = \ll a_1, \dots, a_n \gg$ be a given Pfister form, and $a \in E^\times$. Then q represents $-a$ if and only if $q \otimes \ll a \gg$ is hyperbolic. Thus passing to Galois cohomology and using the Milnor Conjecture, we get the following:

Lemma 8. (Relying on the Milnor Conjectures). *For an n-fold Pfister form $q = \ll a_1, \dots, a_n \gg$ and $a \in E^\times$ the following are equivalent:*

(1) *q represents $-a$.*

(2) $(-a_1) \cup \ldots \cup (-a_n) \cup (-a) = 0$ *in* $\mathrm{H}^{n+1}(G_E, \mathbb{Z}/2)$.

After these introductory remarks we now can announce the characterization of $\mathrm{td}(K|k)$ and of transcendence bases in the case k is a number field as follows. As above, for systems $(t_1, \dots, t_r)$ of elements of $K^\times$, we denote by $q_{(t_1,\dots,t_r)}$ the corresponding Pfister form. Further, for elements $a_i, a, b \in k^\times$ $(i = 1, \dots, r)$ and $(t_1, \dots, t_r)$ given, we also consider the "extended system" $(t_1 - a_1, \dots, t_d - a_d, a, b)$.

Fact 9. *Let $K|k$ be a function field with k a number field. In notations as above the following hold:*

(1) *$K|k$ has transcendence degree d and only if the following conjunction of a $\forall\,\exists$ -sentence <u>and</u> an $\exists\,\forall$ -sentences in the language of fields is true in K:*

" $\forall\, t_1, \dots, t_{d+2}$, the Pfister form $q_{(t_1,\dots,t_{d+2})}$ is universal over $K[\sqrt{-1}]$, <u>and</u> $\exists\, t_1, \dots, t_{d+2}$ such that $q_{(t_1,\dots,t_{d+2})}$ does not represent 0 over $K[\sqrt{-1}]$ ".

(2) *Suppose that* $\mathrm{td}(K|k) = d$. *For a system $(t_1, \dots, t_d)$ of elements of $K^\times$ as above suppose that there exist $a, b \in k^\times$ such that $q_{(t_1,\dots,t_{d+2})}$ does not represent 0 over $K[\sqrt{-1}]$. Then $(t_1, \dots, t_d)$ is a transcendence basis of $K|k$.*

Zusatz: *Let $(t_1, \dots, t_d)$ be a transcendence basis of $K|k$. Then there exist "many" systems of natural numbers $(a_1, \dots, a_d, a, b)$ such that the resulting quadratic form $q_{(t_1-a_1,\dots,t_d-a_d,a,b)}$ is universal, but does not represent* 0 *over $K[\sqrt{-1}]$.*

Proof. (See [P1], Section 1, Fact 1.3 for details.) The idea of the proof is the following: First, by the two Lemmas above it follows that $q_{(t_1,\dots,t_{d+2})}$ is universal over $K[\sqrt{-1}]$. Now let us show that there exist transcendence bases $(t_1, \dots, t_d)$ and natural numbers $(a_1, \dots, a_d, a, b)$ such that the assertion of the Zusatz is true. The proof follows the same pattern as the proof of Fact 5: We set $R = \mathbb{Z}[t_1, \dots, t_d]$, and let S denote the integral closure of R in $K[\sqrt{-1}]$. Then $R \hookrightarrow S$ is a finite ring extension, which is étale on a Zariski open subset of $\mathrm{Spec}\, R \cong \mathbb{A}^d$. In particular, all sufficiently large prime numbers p are not ramified in S. Then using the Generalized Chebotarev Density Theorem, see e.g., Serre [S2], there exist "many" closed points $x \in \mathrm{Spec}\, R$ such that, denoting by

$p = p_x$ the prime number over which x lies, and choosing some natural number b such that $\sqrt{b}$ is not in $K[\sqrt{-1}]$ we have:

i) b is not a square in the residue field $\kappa(x)$.

ii) $R \hookrightarrow S$ is totally split over x.

Now let $\mathfrak{m}$ be the maximal ideal of R defining x, and $\mathfrak{n} \subset S$ a maximal ideal defining a point y of $\operatorname{Spec} S$ over x. Then by the choice of x, it follows that $\mathfrak{m} = (t_1 - a_1, \dots, t_d - a_d, a)$ for some natural numbers a_i $(i = 1, \dots, d)$, and $a := p$ the given prime number; and further, $\mathfrak{n} = (t_1 - a_1, \dots, t_d - a_d, a)$ too; and $(t_1 - a_1, \dots, t_d - a_d, a)$ is a system of regular parameters of the regular ring $S_{\mathfrak{n}}$. In particular, $K[\sqrt{-1}]$ is k-embeddable into $\Lambda = k_{\mathfrak{p}}((t_1 - a_1)) \dots ((t_d - a_d))$ (WHY? **Hint:** Use Hensel's Lemma...), and second, b is not a square in Λ.

Thus making induction on $d \geq 0$, by Lemma 4 we have: $q_{(t_1 - a_1, \dots, t_d - a_d, a, b)}$ does not represent 0 over Λ, thus over K.

Finally, one concludes the proof of Fact 9 in the same way as that of Fact 5.

Summarizing, we have the following way to describe the transcendence degree and even more, transcendence bases of a function field.

Theorem 10. *Let $K|k$ an arithmetic / geometric function field. To systems $(t_1, \dots, t_r)$ of elements of $K^\times$, and rational prime numbers p, let*

$$q^{(p)}_{(t_1, \dots, t_r)} = \textstyle\sum_{\underline{i}} \underline{t}^{\underline{i}} X^p_{\underline{i}}$$

be the homogeneous form over K whose coefficients are all the monomials of the form $\underline{t}^{\underline{i}} = t_1^{i_1} \dots t_r^{i_r}$ with $0 \leq i_1, \dots, i_r < p$. Thus $q_{(t_1, \dots, t_r)} := q^{(2)}_{(t_1, \dots, t_r)}$ is the r-fold Pfister form defined by $(t_1, \dots, t_r)$. Then one has:

(1) *Suppose k has characteristic $\operatorname{char}(k) = p$. Then $(t_1, \dots, t_d)$ is a separable transcendence basis of $K|k$ if and only if $q^{(p)}_{(t_1, \dots, t_d)}$ is universal, but does not represent 0 over K.*

(2) *Let k be algebraically closed with $\operatorname{char}(k) \neq 2$. Then $\operatorname{td}(K|k) = d$ if and only if $q_{(t_1, \dots, t_d)}$ is universal over K for every $(t_1, \dots, t_d)$, and there exist $(t_1, \dots, t_d)$ such that $q_{(t_1, \dots, t_d)}$ does not represent 0 over K.*

If $(t_1, \dots, t_d)$ is a separable transcendence basis of $K|k$, then "for almost all" $(a_1, \dots, a_d)$ with $a_i \in k$ one has: $q_{(t_1 - a_1, \dots, t_d - a_d)}$ is universal over K, but does not represent 0 over K.

(3) *Suppose k is a number field. Then $\operatorname{td}(K|k) = d$ iff $q_{(t_1, \dots, t_{d+2})}$ is universal over $K[\sqrt{-1}]$ for every $(t_1, \dots, t_{d+2})$, but there exist $(t_1, \dots, t_{d+2})$ such that $q_{(t_1, \dots, t_{d+2})}$ does not represent 0 over $K[\sqrt{-1}]$.*

If $(t_1, \dots, t_d)$ is a transcendence basis of $K|k$, then there exist "many" tuples $(a_1, \dots, a_d, a, b)$ of rational numbers such that $q_{(t_1 - a_1, \dots, t_d - a_d, a, b)}$ does not represent 0 over $K[\sqrt{-1}]$.

Generalities about function fields of general type

Let $K|k$ be a function field. We will say that $K|k$ is a *function field of general type,* if $K|k$ has a model $X \to k$ which is a projective smooth k-variety of *general type.* This means the following: Let Ω^1_X be the sheaf of Kähler k-differentials of $X \to k$. Since $X \to k$ is smooth, it follows that Ω^1_X is a locally free $\mathcal{O}_X$-sheaf of rank equal to $\dim(X) = \mathrm{td}(K|k)$. Thus the d^{th} exterior power $\omega_X = \wedge^d \Omega^1_X$ is a locally free $\mathcal{O}_X$-sheaf of rank 1; thus ω_X is a line bundle on X, whose isomorphism class is called the *canonical class* of X.

For every power ω_X^n let $\jmath_n : X \dashrightarrow \mathbb{P}^N_k$ be the canonical k-rational map defined by the global sections of the line bundle ω_X^n, see e.g. [I], Ch. 5. The dimension of the image of $\jmath_n$ as well as the isomorphy type of the (closure of the) image of j_n stabilizes for $n >> 0$, see e.g., loc.cit., Ch. 10. The variety $X \to k$ is said to be of *general type,* if $\jmath_n$ is birational onto its image for $n >> 0$. One has the following:

1) Being a k-variety of general type is a birational notion. In particular, if a function field $K|k$ is of general type, then all its projective smooth models $X \to k$ are k-varieties of general type.

2) Examples:

a) A connected projective smooth curve $X \to k$ is a k-variety of general type if and only if its genus g satisfies $g > 1$.

b) As in the case of curves, there is a finite list of surfaces which are not of general type, see e.g. [Ha], Ch 5, § 6.

One of the significant property of the k-varieties of general type is the following generalization of a well known fact for curves of general type (following from the Hurwitz genus formula):

Theorem 11. *Let $X \to k$ be a k-variety of general type, and suppose that $f : X \dashrightarrow X$ is a dominant rational map. Then f is an isomorphism.*

In particular, if $K|k$ is a function field of general type, then every field k-embedding $K \hookrightarrow K$ is actually an isomorphism of fields.

For a proof see Iitaka [I], Ch.5, § 5.4.

Proof of Theorem A (Lecture 1); see [P1], Section 2.

In the context of Theorem A, we first remark that the fact that K and L are elementarily equivalent implies that k and l are isomorphic finite extensions of their common prime field, which we denote by $\mathbf{k}$. Next, by Theorem 10, the assertion $\mathrm{td}(K|k) = d$ is equivalent to a sentence in the language of fields. Since K and L are elementarily equivalent, it follows that $\mathrm{td}(L|l) = d$.

Let $(t_1, \dots, t_d)$ be a separable transcendence basis of $K|k$, and $x \in K$ a generating element of K over $\mathbf{k}(t_1, \dots, t_d)$. Setting $\underline{X} = (X_1, \dots, X_{d+1})$, let $P(\underline{X}) \in k[\underline{X}]$ be an irreducible polynomial with $P(t_1, \dots, t_d, x) = 0$ in K. In particular, K is then canonically isomorphic to the function field of the affine, irreducible $\mathbf{k}$-variety $X = \operatorname{Spec} \mathbf{k}[\underline{X}]/(P)$. Finally, if k is a number field, then we choose $(t_1, \dots, t_d)$ in such a way that for some natural numbers a, b, the resulting $q_{(t_1,\dots,t_d,a,b)}$ does not represent 0 over $K[\sqrt{-1}]$. This is possible by Fact 9.

Viewing $\underline{\mathfrak{x}} = (\mathfrak{x}_1, \dots, \mathfrak{x}_{d+1})$ as parameters over every field extension $\Lambda|\mathbf{k}$ of $\mathbf{k}$, we consider the following sentence in the language of fields:

$\Psi(\underline{\mathfrak{x}})$: $\exists\, \underline{\mathfrak{x}}$ *such that* $P(\underline{\mathfrak{x}}) = 0$, *and such that:*

- *If* $\mathbf{k} = \mathbb{F}_p$, *then* $q^{(p)}_{(\mathfrak{x}_1,\dots,\mathfrak{x}_d)}$ *is universal but does not represent* 0 *over* Λ.
- *If* $\mathbf{k} = \mathbb{Q}$, *then* $q_{(\mathfrak{x}_1,\dots,\mathfrak{x}_d,a,b)}$ *is universal but does not represent* 0 *over* $\Lambda[\sqrt{-1}]$.

By the choice of $(t_1, \dots, t_d)$ and x, it follows that $\Psi(t_1, \dots, t_d, x)$ is true in K. Since K and L are elementarily equivalent, it follows that $\Psi(\underline{\mathfrak{x}})$ is true in L. In other words, $\exists\, u_1, \dots, u_d, y \in L$ such that $\Psi(u_1, \dots, u_d, y)$ is true in L. Thus $P(u_1, \dots, u_d, y) = 0$ in L, and $\mathcal{C} = (u_1, \dots, u_d)$ is a separable transcendence basis of $L|l$. Thus the mapping

$$x \mapsto y,\ t_i \mapsto u_i\ (1 \le i \le d)$$

defines a field embedding $\imath : K \hookrightarrow L$. By symmetry, it follows that one also has field embeddings $\imath' : L \hookrightarrow K$.

Now suppose that K is of general type. In the notations from above, consider the composition $\jmath = \imath' \circ \imath : K \to K$, which is a field embedding of K into itself. Clearly, $\jmath$ maps k isomorphically onto itself. Since k is either finite or a number field, k has only finitely many automorphisms. Therefore, there exists some $n > 0$ such that the n^{th} iterate $\jmath_n := \jmath \circ \dots \circ \jmath$ of $\jmath$ is the identity on k. In other words, $\jmath_n : K \hookrightarrow K$ is a k-embedding. Since $K|k$ is a function field of general type, it follows that $\jmath_n$ is an isomorphism, see Theorem 11 above, i.e., Iitaka [I], Ch.5, § 5.4. From this it follows that $\jmath$ is a field isomorphism, and finally $\imath$ is a field isomorphism too.

Theorem A is proved.

Lecture 4: On the geometric case and beyond

In this lecture we will sketch the proof of Theorem B (Lecture 1). The details can be found in [P1], Section 3. We will manly stress on the model theoretic aspects of the problem.

First approximation

To begin with, we remark that since K and L are elementarily equivalent, they have isomorphic absolute subfields. In particular, both k and l are algebraically closed fields having the same prime field, which we again denote by $\mathbf{k}$. Moreover, by Theorem 10 above, $\mathrm{td}(K|k) = d = \mathrm{td}(L|l)$.

Next, let us try to do the same as in the arithmetic case, and then explain the new difficulties which arise.

Thus let $(t_1, \dots, t_d)$ be a separable transcendence basis of $K|k$. Now the first observation is that the function field $K|k$ cannot be defined in an "absolute way", i.e., K is in general not algebraic over $\mathbf{k}(t_1, \dots, t_d)$. Let us choose $x \in K$ a generating element of K over $k(t_1, \dots, t_d)$, and let $P(\underline{X}) \in k[\underline{X}]$ be an irreducible polynomial in the variables $\underline{X} = (X_1, \dots, X_{d+1})$ such that $P(t_1, \dots, t_d, x) = 0$ in K. In particular, K is the function field of the affine absolutely irreducible variety $X = \operatorname{Spec} k[\underline{X}]/(P)$.

The polynomial $P(\underline{X})$ itself is defined over a finitely generated $\mathbf{k}$-subalgebra $R = \mathbf{k}[\underline{\alpha}] \subset k$ of k, where $\underline{\alpha} = (\alpha_1, \dots, \alpha_\nu)$ is a system of elements of k. Set $R = \mathbf{k}[\underline{Z}]/(\underline{f})$ with $\underline{f} = (f_1, \dots, f_\mu)$ a system of polynomials in the variables $\underline{Z} = (Z_1, \dots, Z_\nu)$ generating the relation ideal of $\underline{\alpha}$ over $\mathbf{k}$. (Hence the ideal $\mathfrak{p} = (\underline{f}) \subset \mathbf{k}[\underline{Z}]$ is a prime ideal.) In particular, we have $P(\underline{X}) = \mathcal{P}(\underline{Z}, \underline{X}) \bmod \mathfrak{p}$ for some irreducible representative $\mathcal{P} \in \mathbf{k}[\underline{Z}, \underline{X}]$ of P. Denoting by $k_0 = \mathrm{Quot}(R)$ the fraction field of R, and setting $K_0 = k_0(u_1, \dots, u_d, x)$, we have $K = K_0\, k$. In other words, $K|k$ is obtained from $K_0|k_0$ by "extending the constants" from k_0 to k.

Finally, viewing $\underline{\xi} = (\xi_1, \dots, \xi_\nu)$ and $\underline{\mathfrak{x}} = (\mathfrak{x}_1, \dots, \mathfrak{x}_{d+1})$ as parameters over $\mathbf{k}$, we have the following sentence in the language of fields which is true in K and comes close to describing at least $K_0|k_0$, if not the function field $K|k$ itself.

$\Psi_0(\underline{\xi}, \underline{\mathfrak{x}})$: $\exists \underline{\xi}\ \exists \underline{\mathfrak{x}}$ *such that* $\underline{f}(\underline{\xi}) = 0$*, and* $\mathcal{P}(\underline{\xi}, \underline{\mathfrak{x}}) = 0$*, and such that:*

- *If* $\mathbf{k} = \mathbb{F}_p$*, then* $q^{(p)}_{(\mathfrak{x}_1, \dots, \mathfrak{x}_d)}$ *is universal but does not represent* 0 *over* K*.*
- *If* $\mathbf{k} = \mathbb{Q}$*, then* $q_{(\mathfrak{x}_1, \dots, \mathfrak{x}_d)}$ *is universal but does not represent* 0 *over* K*.*

Clearly, setting $\underline{\xi} = \underline{\alpha}$ and $\underline{\mathfrak{x}} = (t_1, \dots, t_d, x)$, we see that $\Psi_0(\underline{\xi}, \underline{\mathfrak{x}})$ is true in K. Since K and L are elementarily equivalent, $\Psi(\underline{\xi}, \underline{\mathfrak{x}})$ is true in L. Hence there exist systems of elements $\underline{\beta} = (\beta_1, \dots, \beta_\mu)$ and $(u_1, \dots, u_d, y)$ of L such that $\Psi_0(\underline{\beta}, u_1, \dots, u_d, y)$ is true in L. The interpretation of this fact is then as follows: There is a ring homomorphism

$$(*) \qquad \phi_{\underline{\beta}, u_1, \dots, u_d, y} : \mathbf{k}[\underline{\alpha}, t_1, \dots, t_d, x] \to \mathbf{k}[\underline{\beta}, u_1, \dots, u_d, y] \subset L$$

which is defined by $\underline{\alpha} \mapsto \underline{\beta}$, $(t_1, \dots, t_d, x) \mapsto (u_1, \dots, u_d, y)$, such that the resulting $\mathcal{C} = (u_1, \dots, u_d)$ is a separable transcendence basis of $L|l$.

Nevertheless, if we now want to use the homomorphism $\phi_{\underline{\beta},u_1,\dots,u_d,y}$ in order to compare $K|k$ and $L|l$, more precisely, to *define a function field embedding*

$$(**) \qquad K_0|k_0 \hookrightarrow L\,|\,l\,,$$

then we run into difficulties. Indeed, $\phi_{\underline{\beta},u_1,\dots,u_d,y}$ defines an embedding of function fields $(**)$ if an only if the following conditions are satisfied:

I) $\phi_{\underline{\beta},u_1,\dots,u_d,y}(R) \subset l$.

II) $\phi_{\underline{\beta},u_1,\dots,u_d,y}$ is injective.

Now the point is that condition I) can be satisfied, the reason for this being the fact that the constant fields k and l *are definable by a predicate* inside K, respectively L. Thus by replacing $\Psi_0(\underline{\xi},\underline{\mathfrak{x}})$ by a more complicated sentence (which is nevertheless very explicit and has a geometrical interpretation), the resulting ring homomorphism $\phi_{\underline{\beta},u_1,\dots,u_d,y}$ satisfies condition I). Nevertheless, the situation with satisfying condition II) is much worse. It is easy to reduce the infectivity of $\phi_{\underline{\beta},u_1,\dots,u_d,y}$ to the one of its restriction to R. But I $***$ **do not know** $***$ at the moment whether in the general case (i.e., without further hypotheses on $K|k$, e.g., $K|k$ is of general type) there do indeed exist $\underline{\beta}, u_1, \dots, u_d, y$ such that $\phi_{\underline{\beta},u_1,\dots,u_d,y}$ is injective.

As explained in [P1], Section 3, B), a substitute for this is the *existence of enough approximations* of $K|k$ with values in $L|l$, see Theorem 3.3, loc.cit.. In particular, in the case $K|k$ is of general type, the existence of enough approximations *a posteriori* implies the existence of $\underline{\beta}, u_1, \dots, u_d, y$ as above such that the resulting $\phi_{\underline{\beta},u_1,\dots,u_d,y}$ is injective.

Tackling I): *The constants are definable*

The aim of this subsection is to show that given a function field $K|k$ with k an algebraically closed field, the constant subfield k of K is definable inside K by a formula $\Xi(\mathfrak{z})$ with one parameter $\mathfrak{z}$, i.e.,

$$k = \{\, a \in K \mid \Xi(a) \text{ is true in } K\,\}$$

To begin with, for arbitrary finite subsets $S \subset k$ having odd cardinality $|S| = 2n+1$, we consider the polynomial $P_S(t) = \prod_{a\in S}(t-a) \in k[t]$, and further $p_{S,t}(T) \in k[t,T]$ defined by:

a) $p_{S,t}(T) = T^2 - P_S(t)$, if $\mathrm{char}(k) \neq 2$.

b) $p_{S,t}(T) = T^2 - T - P_S(t)$, if $\mathrm{char}(k) = 2$.

The main technical point (which, on the other hand, might be well known to specialists, but we cannot give a reference) is the following:

Lemma 12. *Let $K|k$ be a function field with k algebraically closed. Then there exists a bound $c_{K|k}$ with the property: If for some $x \in K$ there exists a* *subset $S = S_x \subset k$ of odd cardinality $|S| > c_{K|k}$ such that $p_{S,x}(T)$ has a root in K, then $x \in k$.*

Proof of Lemma 12. We begin with the following observation: Let $\mathbb{P}^1_t$ be the t-projective line over k, and $K_0 = k(t)$ its function field. For a finite subset $S \subset k$ of odd cardinality $|S| = 2n+1$ as above, let $K_S|K_0$ be the function field extension generated by a root of $p_{S,t}(T)$. Let $C_S \to \mathbb{P}^1_t$ be the normalization of $\mathbb{P}^1_t$ in the Galois field extension $K_S \mid K_0$. It is clear that $C_S \to \mathbb{P}^1_t$ is ramified exactly over $S \cup \{\infty\}$ in case a), respectively in $t = \infty$ in case b). Using the Hurwitz genus formula, we see that the genus g_S of C_S is given by $g_S = n$, thus it depends only on $n = \frac{1}{2}(|S| - 1)$ and not the finite set S itself.

Next let $X \to k$ be a projective normal model of $K|k$, and $\imath : X \hookrightarrow \mathbb{P}^N_k$ be a projective embedding of X. For every $(d-1)$ hyper-planes H_i in "general position" in $\mathbb{P}^N_k$, let $C = X \cap_i H_i$ be the resulting "generic" curve in X. It is well known that the following hold:

- $C \to k$ is a projective normal curve.
- The set of generic points η_C of all the generic curves C is dense in X.
- The genus g of C is independent of the concrete choice of the hyper-planes, it being an invariant of the projective embedding $\imath$.

We will show that we can take $c_{K|k} = 2g + 1$.

Indeed, let $x \in K$ be a non-constant function, and let $S \subset k$ be a finite subset such that $p_{S,x}(T)$ has a root in K. Equivalently, K_S has a $k(x)$-embedding in K. On the other hand, every such embedding is defined by a dominant rational k-map $f : X \dashrightarrow C_S$.

First suppose $\mathrm{td}(K|k) = 1$. Then X is a projective normal curve, and g is the genus of X. The existence of dominant rational k-maps $f : X \dashrightarrow C_S$ as above implies (by the Hurwitz genus formula) that $g \geq g_S$. Thus finally we have $g \geq g_S = \frac{1}{2}(|S| - 1)$, as claimed.

Coming to the general case, since the set of the generic points η_C is dense in X, it follows that there exist points η_C at which $f : X \dashrightarrow C_S$ is defined. Now if f is defined at η_C, then f defines a dominant rational k-map $f : C \dashrightarrow C_S$ of projective normal curves. One concludes as above.

The proof of the Lemma 12 is finished.

Corollary 13. *Let $K|k$ be a function field with k algebraically closed, and $c_{K|k}$ the constant introduced in Lemma 11. Let $S \subset k$ be a fixed finite set of absolute algebraic elements of cardinality $|S| = 2n + 1 > c_{K|k}$. For instance, one could take for S the set of all m^{th} roots of unity for some sufficiently large odd number m. Consider the following formula in one parameter $\mathfrak{z}$ in the language of fields:*

$$\Xi_S(\mathfrak{z}) : \ \exists\, z \text{ such that } p_{S,\mathfrak{z}}(z) = 0.$$

Then $\Xi_S(\mathfrak{z})$ *defines* k *inside* K*, i.e.,* $k = \{\, \alpha \in K \mid \Xi_S(\alpha)$ *is true in* $K\,\}$.

Proof. Clear.

Remark 14. Using Corollary 13, we can tackle the condition I) from the previous subsection as follows: For the given elementarily equivalent function fields $K|k$ and $L|l$ having $\mathbf{k}$ as prime field, choose first $S \subset \mathbf{k}^{\mathrm{alg}}$ finite and of odd cardinality such that $|S| > c_{K|k}, c_{L|l}$. Next replace (the beginning of) the assertion Ψ_0 from the previous subsection as follows:

$\Psi_1(\underline{\xi}, \underline{\mathfrak{x}})$: $\exists \underline{\xi}\ \exists \underline{\mathfrak{x}}$ *such that* $\underline{f}(\underline{\xi}) = 0$*, and* $\Xi_S(\xi_j)$ *(all* $j \le \nu$*), and* $\mathcal{P}(\underline{\xi}, \underline{\mathfrak{x}}) = 0$*, and we have:*

- *If* $\mathbf{k} = \mathbb{F}_p$*, then* $q^{(p)}_{(\mathfrak{x}_1,\dots,\mathfrak{x}_d)}$ *is universal but does not represent* 0 *over* K.
- *If* $\mathbf{k} = \mathbb{Q}$*, then* $q_{(\mathfrak{x}_1,\dots,\mathfrak{x}_d)}$ *is universal but does not represent* 0 *over* K.

Clearly, Ψ_1 is true in $K|k$, thus it is true in L. Hence in the notations from subsection G), the resulting system $\underline{b}$ of elements of L consists actually of elements of l. Thus the resulting ring homomorphism $\phi_{\underline{b},u_1,\dots,u_d,y}$, as constructed at the end of subsection G) above, maps R into l.

Tackling II): *Approximations of function fields...*

Actually I am not going to explain the notions here, see loc.cit. for the quite technical stuff... But in order to simplify it a little bit one might try the following

Problem:

Let R be an algebra of finite type over some prime field, k the algebraic closure of Quot(R). Let further $\mathcal{X} \to R$ and $\mathcal{Y} \to R$ be (projective, smooth) R-varieties. We denote by $X \to k$ and $Y \to k$ the geometric generic fibers of $\mathcal{X} \to R$, respectively $\mathcal{Y} \to R$. And for any point $s \in \operatorname{Spec} R$, let $X_s \to \kappa_s$ and $Y_s \to \kappa_s$ the corresponding fibers. Next let $\Sigma_{\mathcal{X}} \subset S$ to be the set of all points of $\operatorname{Spec} R$ having the property: There exists an embedding $\imath_s : \kappa_s \hookrightarrow k$ which "prolongs" to a dominant separable rational map $Y \to X_s$. This means that denoting by $X_{\imath_s} = X_s \times_\imath k$ the base change of X_s to k via $\imath$, one further has a dominant k-rational map $\jmath_s : Y \to X_{\imath_s}$ whose generic fiber is a separable field homomorphism.

Suppose that both $\Sigma_{\mathcal{X}}$ and $\Sigma_{\mathcal{Y}}$ are dense in $\operatorname{Spec} R$.

Prove/Disprove: *There exists an isomorphism* $k \hookrightarrow k$ *which "prolongs" to a dominant separable morphism of schemes* $Y \to X$*, and conversely.*

We remark that from the hypothesis above, one gets $\dim_k(X) = \dim_k(Y)$. Also, in the case $X \to k$ is a variety of general type, the Problem above has a positive answer (well, this follows from the "approximations of function fields" of loc.cit.)... Finally, a positive answer to the Problem would make the assertion of Theorem B much stronger, in the sense that the first part of point(2) would hold in the same form as in Theorem A, thus without the hypothesis " $K|k$ is a function field of general type".

Questions:

The main question/problem here is to either prove or disprove the EeVIP in the case $K|k$ is an arithmetic and/or geometric function field of non-general type. As a first approximation to the general question, the following problems could give more evidence in the favor of the EeVIP.

Q1: *Is the hypothesis "without CM" in* DURET*'s* [D2]*, and* PIERCE*'s* [Pi] *result, in the case $K|k$ is the function field of an elliptic curve, necessary?*

More generally, let $K|k$ be the function field of a projective smooth curve X. Suppose that the genus g of X satisfies $g = 1$ or $g = 0$. In the first case X is a homogeneous space (form short, a form) of an elliptic curve over k, see e.g., [Sl], especially Ch X, §2–3; and in the second case, X is a form of the projective line over k, i.e., a one dimensional Brauer–Severi variety over k, see [S4]; especially Ch X, §6.

Q2: *Prove or disprove EeVIP in the arithmetic case if $g = 0$.*

Q3: *Prove or disprove EeVIP in the arithmetic case if $g = 1$.*

Naturally, one can ask the same question in the case $\mathrm{td}(K|k) = 2$, but the situation seems to be much more complicated.

Q4: *Extend the method from* Tackling I) *to "non-mordellic" base fields, like: real / p-adically closed, or a PAC field, or even a "large field"* (see e.g., [P3] for definitions).

Consequence: In Theorem B we could replace the algebraically closed base field k by any field k from above.

Q5: *Consider the questions* Q2, Q3 *as above, if k one of the following types: real or p-adically closed, or PAC, or even a "large field".*

Q6: *Find an "axiomatization" of the isomorphism type of function fields of general type.*

Q∞) *Prove or disprove the EeVIP in general!...*

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