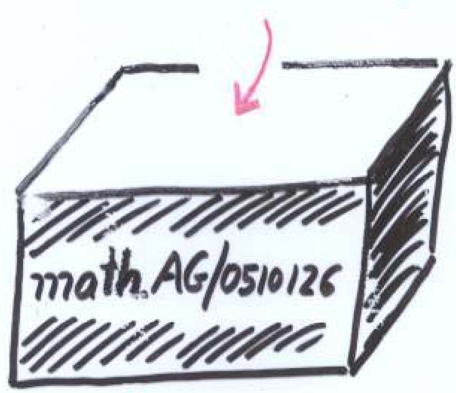
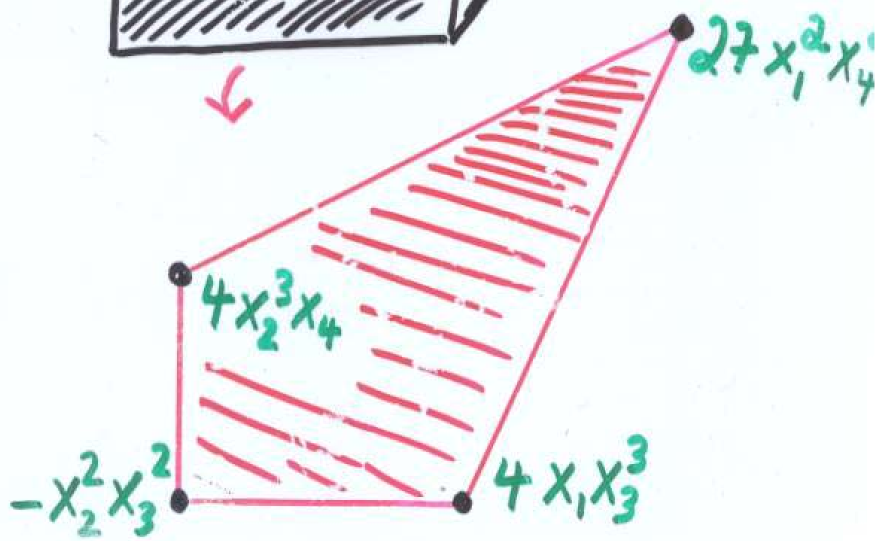
Bernd Sturmfels' Arizona Lecture #3
Tropical Discriminants

Input: $\begin{pmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix}$

black box



Output:



My 2005 Summer Vacation

with Eva and Alicia in Switzerland

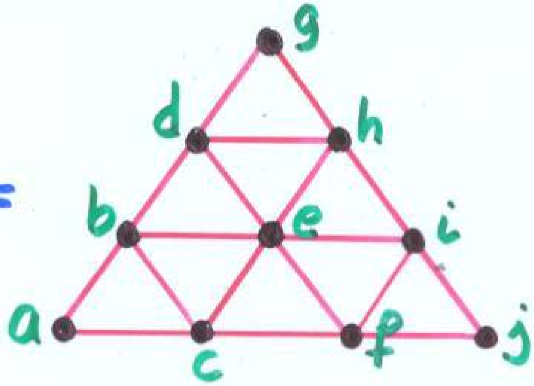
led to ... an explicit ...

Combinatorial Description of the tropicalization of the A-discriminant Δ_A for any integer matrix A.

If $\mathrm{codim}(\Delta_A) = 1$ this gives an efficient method for computing the Newton polytope of Δ

Elliptic Curves Revisited

Input: $A =$



Output: The Newton polytope of Δ_A:

$$\Big\{ (a,b,c,d,e,f,g,h,i,j) \in \mathbb{R}^{10}_{\geq 0} :$$

$$\begin{bmatrix} 3 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 & 0 & 1 & 2 & 3 \end{bmatrix} \cdot \begin{pmatrix} a \\ b \\ \vdots \\ j \end{pmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix},$$

$$2a+b+c \geq 2, \quad 2j+f+i \geq 2, \quad 2g+d+h \geq 2,$$

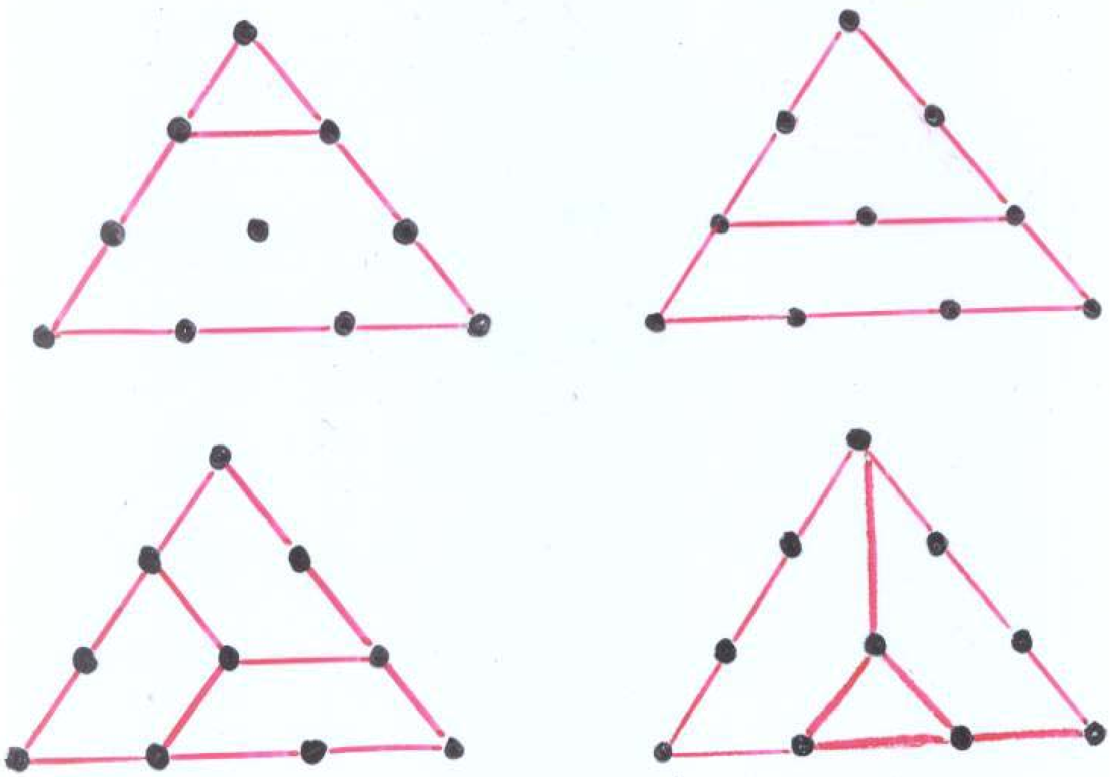
$$b+d+e \leq 9, \quad e+h+i \leq 9, \quad c+e+f \leq 9$$

$$2a+b+c+d+g \geq 3, \quad 2g+d+h+i+j \geq 3 \Big\}$$

This 7-diml. polytope has f-vector

(133, 513, 846, 764, 402, 120, 18

The 18 facets come in 4 classes corresponding to the following coarsest subdivisions of A :



Tropical Horn Uniformization

$\ker A$ is a linear variety in $\mathbb{P}^{n-1}_{\mathbb{C}}$

Its tropicalization $\mathcal{T}(\ker A)$ can be computed from the matroid of A

Theorem: The tropical A-discriminant is the sum of the linear space spanned by the rows of A and the tropical linear space determined by the kernel of A. In symbols

---

$$\mathcal{T}(\Delta_A) = \underbrace{\mathrm{rowspace}(A)}_{d-1} + \underbrace{\mathcal{T}(\ker A)}_{n-d-1}$$

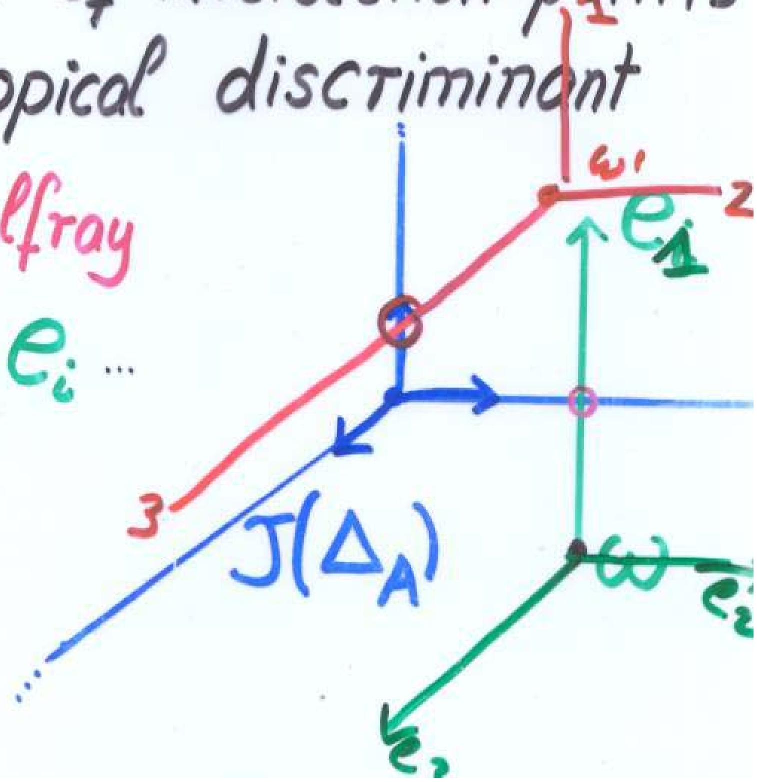
Recovering the Newton Polytope

Suppose Δ_A is a hypersurface.
(The formula [illegible] gives a test for this)

Theorem Fix $\omega \in \mathbb{R}^n$ generic.
The exponent of x_i in $in_\omega(\Delta_A)$ equals
the number of intersection points
of the tropical discriminant
with the halfray

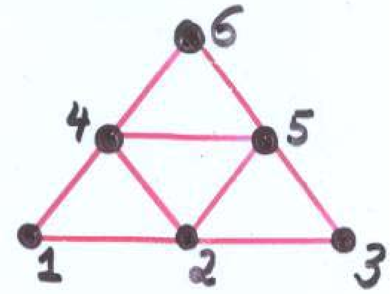
$$\omega + \mathbb{R}_{\geq 0} e_i \ldots$$

... counting multiplicities.

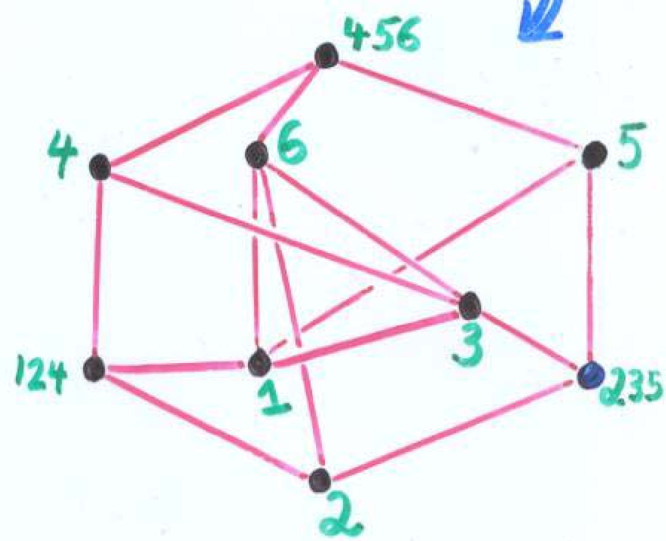


Example (T.H.U) $d=3,\ n=6$

$$A = \begin{pmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{pmatrix}$$

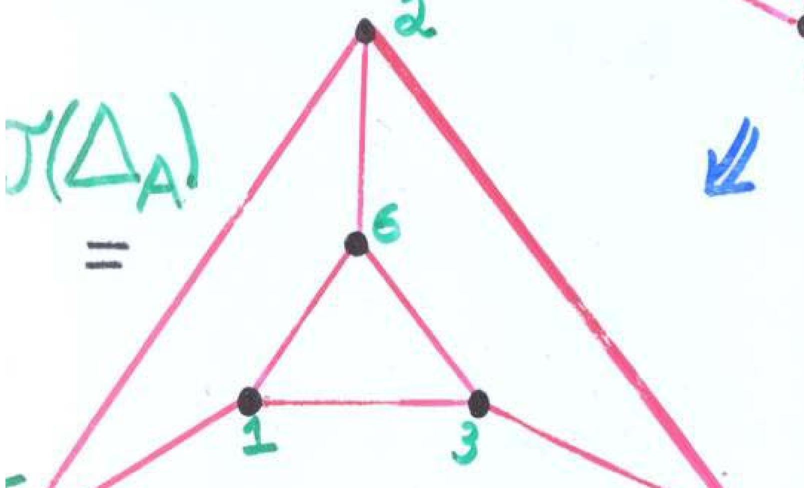


$\mathcal{T}(\mathrm{Ker}\, A) =$



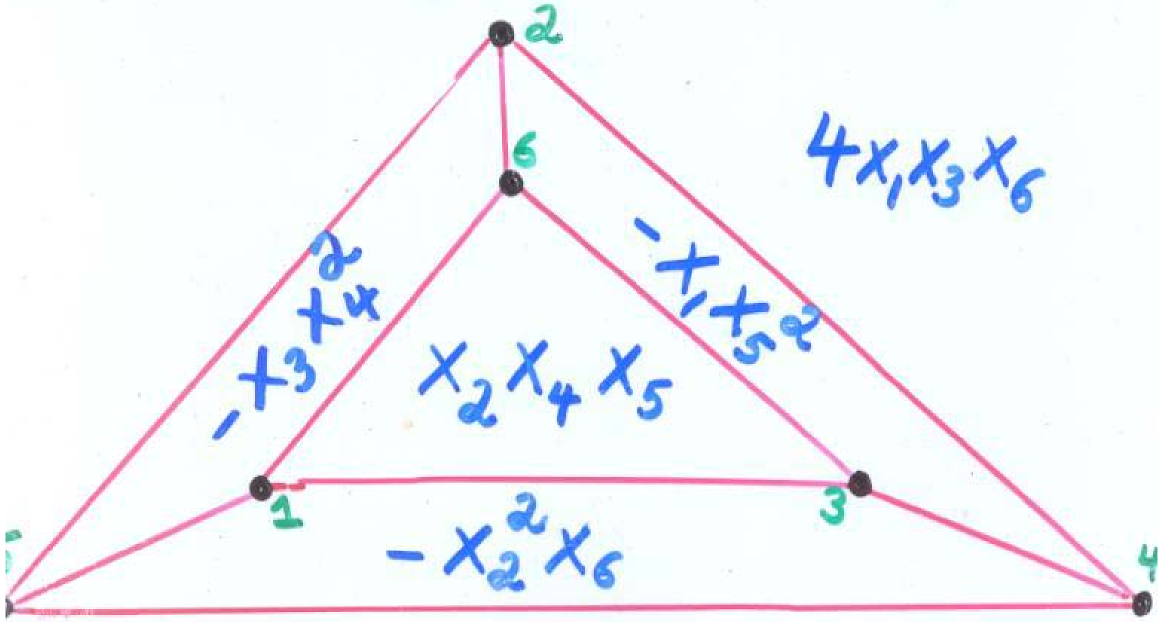
This is a
3-dim'l fan
in 6-dim'l space

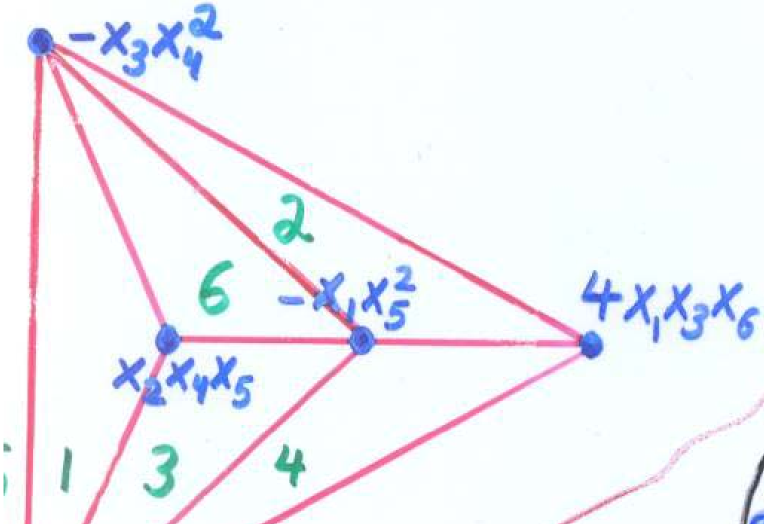
$\mathcal{T}(\Delta_A)$
$=$



This is the
normal fan of
the Newton polytop
of Δ_A

From Toblerone to Bipyramid ... '8





... to symmetric 3x3-determinan

$$\Delta_A = \frac{1}{2}\det\begin{pmatrix} 2x_1 & x_2 & x_4 \\ x_2 & 2x_3 & x_5 \\ x_4 & x_5 & 2x_6 \end{pmatrix}$$

Our running example $\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 2 & 3 & 5 & 7 & 11 & 13 & 17 & 19 \\ 53 & 47 & 43 & 41 & 37 & 31 & 29 & 23 \end{bmatrix}$

is row equivalent to

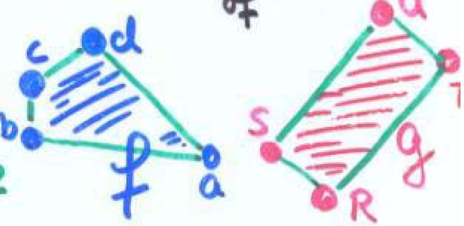
$$A = \begin{array}{c} \begin{matrix} a & b & c & d & R & S & T & U \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 2 & 0 & 0 & 1 & 1 & 0 & 3 & 2 \\ 0 & 1 & 3 & 5 & 0 & 2 & 6 & 8 \end{bmatrix} \end{array}$$

This Cayley matrix represents a system of two equations in two unknowns

$$f(x,y) = ax^2 + by + cy^3 + dxy^5$$
$$g(x,y) = Rx + Sy^2 + Tx^3y^6 + Ux^2y^8$$

For generic coefficients a,b,c,d,R,S,T,U, the system $f(x,y) = g(x,y) = 0$ has 24 solutions $(x,y) \in (\mathbb{C}^*)^2$

mixed area of



The discriminant Δ_A is the irreducible polynomial in a, b, c, d, R, S, T, U which vanishes whenever

$$f(x,y) = g(x,y) = 0$$

has a solution $(x,y) \in (\mathbb{C}^*)^2$ of multiplicity two or more.

Can be computed by adding the equation

$$\frac{\partial f}{\partial x} \cdot \frac{\partial g}{\partial y} - \frac{\partial g}{\partial x} \cdot \frac{\partial f}{\partial y} = 0$$

and eliminating x and y ......

Q1: What is the degree of Δ_A ?

Q2: What is the A-degree of Δ_A ?

Q3: What is the Newton polytope of Δ_A ?

Q4: Why should applied mathematicians care ¿

The Horn Uniformization à la Kapranov

.... is a parametric representation of the A-discriminant

$$a = -2c_4 t_1 t_3^2$$
$$b = (c_2 - 2c_3 + c_4) t_1 t_4$$
$$c = (c_2 + 3c_3) t_1 t_4^3$$
$$d = (-2c_2 - c_3 + c_4) t_1 t_3 t_4^5$$
$$R = (c_1 + c_4)\, t_2 t_3$$
$$S = (-c_1 - c_2 - c_4) t_2 t_4^2$$
$$T = (-c_1 + c_3 + 2c_4) t_2 t_3^3 t_4^6$$
$$U = (c_1 + c_2 - c_3 - 2c_4) t_2 t_3^2 t_4^8$$

Q: How to implicitize this map $\mathbb{C}^8 \to \mathbb{C}^8$?

A: Do it tropically first!

>> GROTHENDIECK in the tropics <<

What the black box produces

The Newton polytope of Δ_A is a 4-dimensional polytope with f-vector $(74, 158, 110, 26)$.

The 74 extreme monomials of Δ_A are

$$a^{10} b^{18} c^{18} d^{1} S^{18} T^{29} U^{2},$$

$$a^{10} b^{18} c^{8} d^{11} S^{22} T^{27},$$

$$\cdots\cdots$$

$$b^{42} c^{2} d^{3} R^{11} S^{2} T^{26} U^{10}.$$

A-degree $\begin{pmatrix} 47 \\ 49 \\ 112 \\ 303 \end{pmatrix}$

The total number of lattice points in this polytope is

$$\underline{21{,}176} = 74 + 81 + 753 + 4082 + 16186$$

»LATTE«

What the black box does

- Start with the 60 triangles representing the 3-dimensional tropical linear space $\mathcal{T}(\ker A)$
- Take its image under the linear map $\mathbb{R}^8 \to \operatorname{coker} A$
- This collapses the 60 cones to 48 immersed cones
- The result is a 3-dim. fan with 158 cones on 26 rays
- This is the tropical hypersurface $\mathcal{T}(\Delta_A)$
- Now reconstruct the Newton polytope ...