Bernd Sturmfels'

Arizona Lecture #4

Tropical Implicitization

oint work with Jenia Tevelev & Josephine Yu

Input:

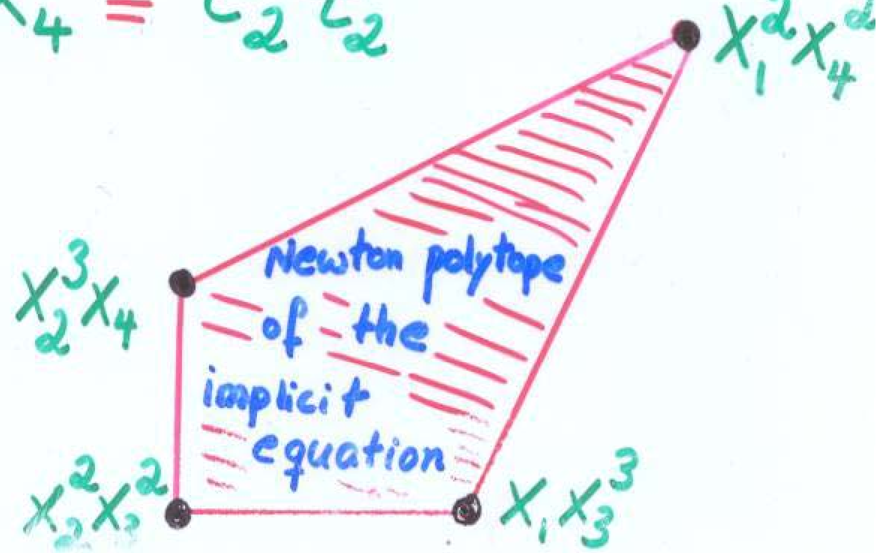
$$x_1 = c_1 t_1^3$$

$$x_2 = (-2c_1 + c_2)\, t_1^2 t_2$$

$$x_3 = (c_1 - 2c_2)\, t_1 t_2^2$$

$$x_4 = c_2 t_2^3$$

Output:



The Problem of Implicitization

Given n polynomials $f_1, \dots, f_n$
in d unknowns $t = (t_1, \dots, t_d)$,
compute the **Kernel** of the ring map

$$\mathbb{C}[x_1, \dots, x_n] \to \mathbb{C}[t_1, \dots, t_d]$$
$$x_i \mapsto f_i(t)$$

This is a prime ideal I in $\mathbb{C}[\bar{x}]$.

This is soooo hard ....
... so instead we do

Tropical Implicitization

Compute the tropical variety $\mathcal{T}(I)$
directly from $f_1, \dots, f_n$

$$X_1 = t_1 t_2 (t_1^4 - t_2^4)$$

$$X_2 = \mathrm{Hessian}(x_1(t))$$

$$X_3 = \mathrm{Jacobian}(x_1(t), x_2(t)$$

The implicit equation for this map $\mathbb{C}^2 \to \mathbb{C}^3$ equals $g(x_1, x_2, x_3) = ???$

Can we recover I from $\mathcal{J}(I)$?

Not quite .... but its Chow polytope

Theorem 2.2. [DFS]

$c = n - d$

Let ω be a generic vector in $\mathbb{R}^n$. A monomial prime $\langle x_{j_1}, \ldots, x_{j_c} \rangle$ is associated to the initial monomial ideal $in_\omega(I)$ if and only if $\mathcal{J}(I)$ meets the cone

$$\omega + \mathbb{R}_{\geq 0}\{e_{j_1}, \ldots, e_{j_c}\}.$$

The number of intersection points, counted appropriately, equals the multiplicity of this prime in $in_\omega(I)$.

Tropical Implicitization of Curves

$d = 1$

Here $f_1(t), f_2(t), \dots, f_n(t)$ are rational functions in one unknown t

Let $\alpha_1, \alpha_2, \dots, \alpha_m \in \mathbb{C} \cup \{\infty\}$ be all poles and zeros. Write

$$f_i(t) = \prod_{j=1}^{m} (t - \alpha_j)^{u_{ij}}$$

The m vectors $(u_{i1}, u_{i2}, \dots, u_{in})$ sum to zero in $\mathbb{R}^n$. The union of their rays equals the tropical curve $\mathcal{T}(I)$

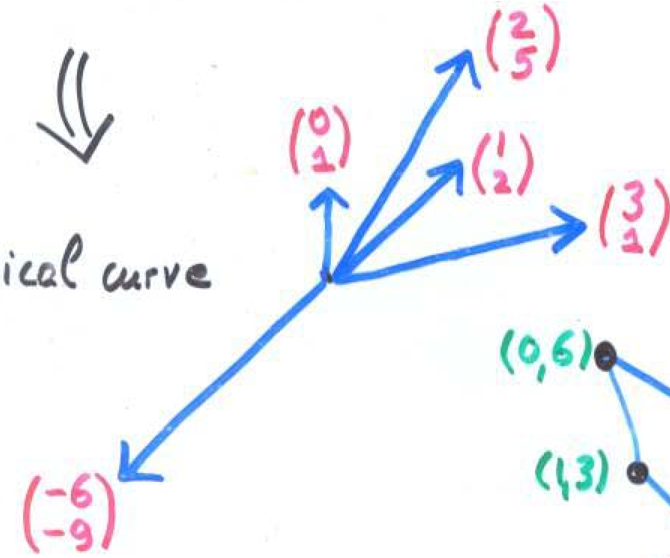
A parametrized plane curve

$$x = t^2(t-1)^1(t-2)^0(t-3)^3$$

$$y = t^5(t-1)^2(t-2)^1(t-3)^1$$

$\Downarrow$

tropical curve



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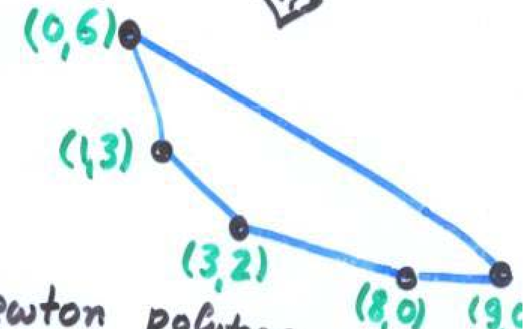


Newton polytope of

$g(x,y) =$

$$x^9 + 4x^8 + 494x^7y - 3x^6y^2 + 1978x^6y + 61214x^5y^2 + \cdots + 51018336xy^3$$

17 terms

How about for $d \geq 2$ unknowns ?

Well, **if** $\bigcup_{i=1}^{n} \{ f_i = 0 \}$

defines a normal crossing divisor with smooth components on some compactification X of $(\mathbb{C}^*)^d$ then a similar construction works ...

[Hacking-Keel-Tevelev '06]

Q: How to make this computational ?

A: Focus on the Newton polytopes of the f_i

Genericity Assumption

Suppose the coefficients of f_i are generic relative to fixing the Newton polytope $P_i = New(f_i)$

Choose an $m \times d$-matrix A and column vectors $b_1, \ldots, b_n \in \mathbb{R}^m$ such that

$$P_i = \{ u \in \mathbb{R}^d : Au \geq b_i \} \text{ for } i = 1, \ldots, n$$

Example "Plane Curves" $(n=2,\ d=1 \Rightarrow m=2)$

$$A = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad b_1 = \begin{bmatrix} \alpha \\ -\beta \end{bmatrix}, \ b_2 = \begin{bmatrix} \gamma \\ -\delta \end{bmatrix}$$

The incidence fan

The incidence fan of $P_1, \dots, P_n$ is the coordinate fan in $\mathbb{R}^{n+m}$ with basis $e_1, \dots, e_n, E_1, \dots, E_m$ whose cones are the orthants

$$\mathbb{R}_{\geq 0}\{e_{i_1}, \dots, e_{i_k}, E_{j_1}, \dots, E_{j_\ell}\}$$

such that the face of

$$P_{i_1} + \cdots + P_{i_k}$$

indexed by $j_1, \dots, j_\ell$ has codimension $\leq \ell$.

For $\ell = 0$ take all proper subsets of $\{e_1, \dots, e_n\}$

Theorem

The tropical variety $\mathcal{T}(I)$ is the image of the incidence fan of $P_1, \dots, P_n$ under the linear map

$$\mathbb{R}^{n+m} \to \mathbb{R}^n$$
$$(y, z) \mapsto y + z \cdot B$$

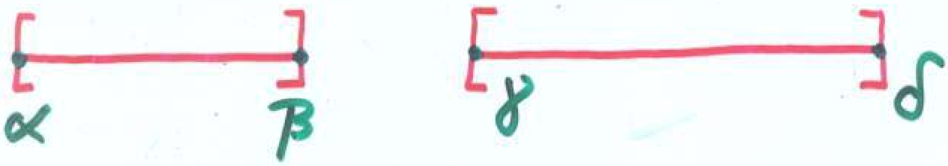
where B is the matrix with columns b_i.

The hypersurface case

If $n = d+1$ and $I = \langle g \rangle$ is principal we get a combinatorial rule for constructing the Newton polytope of g from $P_1, \dots, P_n$

Tropical Implicitization of Plane Curves

Input Two one-dimensional Newton polytopes:



Output The Newton polygon $Q \subset \mathbb{R}^2$ of the implicit equation $g(x,y) = 0$

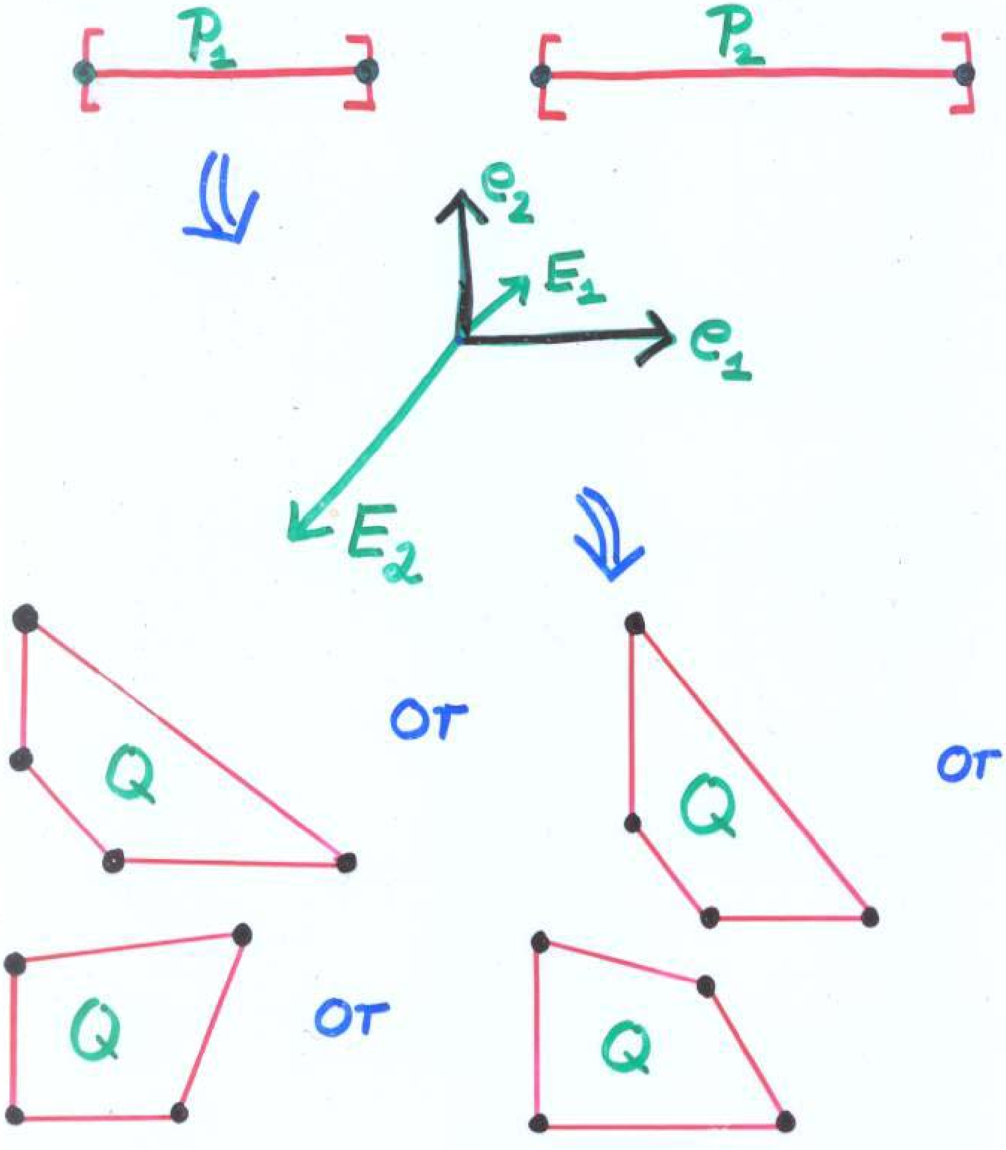
Case 1: If $\alpha \geq 0$ and $\gamma \geq 0$ then
$$Q = \mathrm{conv}\{(0,\beta),(0,\alpha),(\gamma,0),(\delta,0)\}$$

Case 2: If $\beta \leq 0$ and $\delta \leq 0$ then
$$Q = \mathrm{conv}\{(0,-\alpha),(0,-\beta),(-\delta,0),(-\gamma,0)\}$$

Case 3: If $\alpha \leq 0$, $\delta \geq 0$ and $\beta\gamma \geq \alpha\delta$ then
$$Q = \mathrm{conv}\{(0,\beta-\alpha),(0,0),(\delta-\gamma,0),(\delta,-\alpha)\}$$

Case 4: If $\beta \geq 0$, $\gamma \leq 0$ and $\beta\gamma \leq \alpha\delta$ then
$$Q = \mathrm{conv}\{(0,\beta-\alpha),(0,0),(\delta-\gamma,0),(-\gamma,\beta)\}$$

$d=1$, $n=2$ in Pictures



Now, Josephine Yu will show

$d=2$, $n=3$ in Pictures