The p-adic upper half plane

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Introduction

The p-adic upper half plane $\mathcal{X}$ is a rigid analytic variety over a p-adic field K, on which the group $\mathrm{GL}_2(K)$ acts, that Mumford introduced (as a formal scheme) as part of his efforts to generalize Tate's p-adic uniformization of elliptic curves to curves of higher genus. The $\mathbf{C}_p$–valued points of $\mathcal{X}$ are just $\mathbf{P}^1(\mathbf{C}_p) - \mathbf{P}^1(K)$, with $\mathrm{GL}_2(K)$ acting by linear fractional transformations. Mumford showed that the appropriate generalization of Tate's elliptic curves – the "totally split" curves of higher genus – could be constructed as the quotient of the space $\mathcal{X}$ by appropriate discrete groups $\Gamma \in \mathrm{GL}_2(K)$. Mumford's work acquired even greater significance for number theorists when Cerednik and Drinfeld showed that an important class of modular curves – the Shimura curves – could be constructed by p-adic uniformization by choosing for the discrete group Γ an appropriate arithmetic subgroup coming from a definite quaternion algebra over $\mathbf{Q}$. More recently, the p-adic upper half plane has figured prominently in recent developments in arithmetic geometry. In Section 1 of these notes, we will construct the space $\mathcal{X}$ as a rigid variety and describe some of its most fundamental geometric properties, and in subsequent sections we will explore some of this more recent work.

Our focus in Section 2 will be the analytic theory of $\mathcal{X}$, and in particular the relationship between spaces of functions on the p-adic upper half plane and distributions on $\mathbf{P}^1(K)$, which is the "boundary" of $\mathcal{X}$. One main result will be the construction of the Poisson integral for $\mathcal{X}$; in a manner analogous to the classical Poisson transform, this integral allows one to recover rigid analytic functions on $\mathcal{X}$ from appropriate boundary distributions by integrating against a kernel function.

In Sections 3 and 4, we establish connections between number theory and the geometry of the p-adic upper half plane, with particular emphasis on the relationship between the p-adic upper half plane and $\mathcal{L}$-invariants. If E is an elliptic curve over $\mathbf{Q}$ with split multiplicative reduction at p and analytic Mordell-Weil rank zero, then [**25**] conjectured and [**19**] proved that the p-adic L-function of the modular form f corresponding to E vanishes to order 1 and the special value of $L_p'(1)$ differs from the classical special value by the number

$$\mathcal{L}(f) = \frac{\log_p(q_E)}{\mathrm{ord}_p(q_E)}$$

[1] First published in the University Lecture Series 45 (2008), published by the American Mathematical Society. First author supported by NSF Grant DMS-0653023. Second author supported by NSF Grant DMS-0245410.

where q_E is the Tate period of the curve E at p. The paper [**25**] made a weak conjecture (the exceptional zero conjecture) about the relationship between the special values of the p-adic and classical L-functions of higher weight modular forms, and in an attempt to make that conjecture more precise, different mathematicians introduced a whole collection of more general $\mathcal{L}$-invariants associated to such forms. Many of these $\mathcal{L}$-invariants – all of which are now known to be equal – are related in some way to the p-adic upper half plane, and after a general discussion of $\mathcal{L}$-invariants we focus in particular on three such: one defined by the second author of these notes; one defined by Orton; and one defined by Breuil. Much of Section 3 is devoted to Orton's proof of the exceptional zero conjecture using her invariant, while Section 4 discusses Breuil's invariant and its relationship with the cohomology of modular curves and a possible p-adic Langlands correspondence.

1. Geometry of the p-adic upper half plane

1.1. Basic notations. We let K denote a finite extension of the p-adic numbers $\mathbf{Q}_p$, and we let G be the group $\mathrm{GL}_2(K)$. If o_K denotes the ring of integers in K, then we write G_o for the maximal compact subgroup $\mathrm{GL}_2(o_K)$ in G. We let π be a uniformizing parameter for o_K and write $|\cdot|$ for the normalized p-adic absolute value on K extending the p-adic absolute value on $\mathbf{Q}_p$. We will also use the additive valuation

$$\omega : K \to \mathbf{Z}$$

normalized so that $\omega(\pi) = 1$.

Let V be a fixed two dimensional vector space over K, viewed as a space of row vectors, on which G acts on the left by the formula

$$g([x,y]) = [x,y]\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}.$$

When we refer to $\mathbf{P}^1$ we mean specifically $\mathbf{P}(V)$ with its G-action. We let Ξ_0 and Ξ_1 be the dual elements in V^* to the standard basis vectors $[1,0]$ and $[0,1]$ in V; they are "homogeneous coordinates" on $\mathbf{P}^1$. A linear form in Ξ_0 and Ξ_1 is called unimodular if at least one of its two coefficients is a unit in o_K and the other coefficient lies in o_K.

The coordinate function

$$z = \frac{\Xi_0}{\Xi_1}$$

is acted on by a matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ through the formula

$$\begin{aligned} g_*(z)([x,y]) &= z(g^{-1}([x,y])) \\ &= z([x,y]g) \\ &= z([ax+cy, bx+dy]) \\ &= \frac{az+c}{bz+d} \end{aligned}$$

1.2. The p-adic upper half plane. The central object of interest in this series of lectures is the p-adic upper half plane $\mathcal{X}$, a rigid analytic space whose L-points are given by the rule

$$\mathcal{X}(L) = \mathbf{P}^1(L) \backslash \mathbf{P}^1(K)$$

for complete extension fields L of K.

1.2.1. *An admissible covering.* To construct $\mathcal{X}$, we need to describe an admissible covering that defines its rigid structure. We will describe an increasing sequence of affinoid subdomains $\mathcal{X}_n^-$ in $\mathbf{P}^1$, for integer $n \geq 1$, and some related admissible domains $\mathcal{X}_n$, so that $\mathcal{X}$ is the union of the $\mathcal{X}_n^-$ and the $\mathcal{X}_n$. Essentially, $\mathcal{X}_n^-$ is constructed by deleting from $\mathbf{P}^1$ smaller and smaller balls around the rational points.

We will describe this in a coordinate-free way that may not be the simplest approach in dimension 1 but is easier to generalize to higher dimension (see [**35**]).

Given $x \in \mathbf{P}^1(\mathbf{C}_p)$, we may choose homogeneous coordinates $x = [x_0, x_1]$ for x that are unimodular, meaning that both coordinates are integral, but at least one is not divisible by π. For a real number $r > 0$, let

$$B(x,r) = \{y \in \mathbf{P}^1(\mathbf{C}_p) : \omega(y_0x_1 - y_1x_0) \geq r\},$$

where we always take a unimodular representative $[y_0, y_1]$ of y. Also define

$$B^-(x,r) = \{y \in \mathbf{P}^1(\mathbf{C}_p) : \omega(y_0x_1 - y_1x_0) > r\}.$$

LEMMA 1.2.1. *Let x and x' be two elements of $\mathbf{P}^1(K)$, and let n be a positive integer. Then $B(x,n) \cap B(x',n) \neq \emptyset$ if and only if $[x_0, x_1] \equiv \lambda[x_0', x_1'] \pmod{\pi^n}$ for some unit $\lambda \in o_K^*$.*

PROOF. Suppose $y \in B(x,n) \cap B(x',n)$. Then we have the equations:

$$\begin{array}{rcl} \omega(x_1y_0 - y_1x_0) & \geq & n \\ \omega(x_1'y_0 - y_1x_0') & \geq & n. \end{array}$$

Suppose for convenience that y_0 is a unit. Then we can conclude that

$$\omega(x_0'x_1 - x_1'x_0) \geq n.$$

This means that the vectors $[x_0, x_1]$ and $[x_0', x_1']$ are linearly dependent modulo π^n, which is the claim. Conversely, if the vectors are linearly dependent mod π^n, we may construct a y in the intersection of the two sets by choosing a unimodular representative of the kernel of the appropriate matrix made out of x and x'. □

DEFINITION 1.2.2. For each integer $n > 0$, let $\mathcal{P}_n$ be a set of representatives for the points of $\mathbf{P}^1(K)$ modulo π^n. Let $\mathcal{X}_n$ be the set

$$\mathcal{X}_n := \mathbf{P}^1(\mathbf{C}_p) \setminus \bigcup_{x \in \mathcal{P}_n} B(x,n)$$

Let $\mathcal{X}_n^- \subset \mathcal{X}_n$ be the set

$$\mathcal{X}_n^- := \mathbf{P}^1(\mathbf{C}_p) \setminus \bigcup_{x \in \mathcal{P}_n} B^-(x, n-1)$$

Let

$$\mathcal{X} = \bigcup_n \mathcal{X}_n = \bigcup_n \mathcal{X}_n^-.$$

We can make the sets $\mathcal{X}_n$ and $\mathcal{X}_n^-$ more explicit. Fix an integer $n \geq 1$. Then we can choose representatives for $\mathcal{P}_n$ as follows:

$[a_i, 1]$, where $\{a_i\}_{i=0}^{q^n-1}$ is a set of representatives in o_K for $o_K/\pi^n o_K$;

$[1, b_i]$, where $\{b_i\}_{i=0}^{q^{n-1}-1}$ is a set of representatives in πo_K for $\pi o_K/\pi^n o_K$.

Then it follows from the definitions that $\mathcal{X}_n$ is the set of points $x \in \mathbf{P}^1$ defined by the inequalities

$$\begin{aligned} \omega(z(x) - a_i) &< n \qquad i = 0, \ldots, q^n - 1 \\ \omega\left(\frac{1}{z(x)} - b_i\right) &< n \quad i = 0, \ldots, q^{n-1} - 1. \end{aligned}$$

It will be useful later to have slightly different inequalities for the covering domains. In particular, it is easy to check that if $\omega(b) < n$, then

$$\omega(1/z - b) < n \Leftrightarrow \omega(z - 1/b) < n - 2\omega(b).$$

Consequently we can (choosing $b_0 = 0$) rewrite the system of inequalities defining $\mathcal{X}_n$ as

$$\begin{aligned} \omega(z - a_i) &< n && \text{for } i = 0, \ldots, q^n - 1 \\ \omega(z - 1/b_i) &< n - 2\omega(b_i) && \text{for } i = 1, \ldots, q^{n-1} - 1 \\ \omega(z) &> -n && \end{aligned} \tag{1.2.3}$$

and, for $\mathcal{X}_n^-$,

$$\begin{aligned} \omega(z - a_i) &\leq n - 1 && \text{for } i = 0, \ldots, q^n - 1 \\ \omega(z - 1/b_i) &\leq n - 1 - 2\omega(b_i) && \text{for } i = 1, \ldots, q^{n-1} - 1 \\ \omega(z) &\geq 1 - n && \end{aligned} \tag{1.2.4}$$

Proposition 1.2.5. *$\mathcal{X}$ is an admissible open subdomain of $\mathbf{P}^1$ and the coverings of $\mathcal{X}$ by the families $\{\mathcal{X}_n\}_{n=1}^{\infty}$ and $\{\mathcal{X}_n^-\}_{n=1}^{\infty}$ are admissible coverings. In the latter case, the covering is by open affinoid domains.*

Proof. See the discussion following Lemma 3 in [**35**]. □

1.2.2. *The ring $\mathcal{O}_{\mathcal{X}}$ of entire functions on $\mathcal{X}$.* The ring of entire functions on $\mathcal{O}_{\mathcal{X}}$ is the projective limit of the affinoid algebras $\mathcal{O}(\mathcal{X}_n^-)$ as $n \to \infty$:

$$\mathcal{O}_{\mathcal{X}} := \varprojlim_n \mathcal{O}(\mathcal{X}_n^-).$$

Many important function-theoretic properties of $\mathcal{X}$ flow from two key facts:

(1) $\mathcal{X}$ is a (smooth, one-dimensional) rigid analytic Stein space;
(2) the restriction maps between the affinoid algebras $\mathcal{O}(\mathcal{X}_n^-)$ are compact maps. (Recall that a continuous linear map $f : A \to B$ between Banach spaces is called compact if the image of the unit ball in A has compact closure in B).

The compactness property (2) of the transition maps is a fairly general phenomenon. At its core is the following special case. Consider the affinoid ball of points z with $\omega(z) \geq -1$, with its associated affinoid algebra $K\langle \pi T \rangle$. Consider also the restriction map to the sub-affinoid of points z with $\omega(z) \geq 0$, and its affinoid algebra $K\langle T \rangle$. Ignoring the ring structure, we see that the image in $K\langle T \rangle$ of the unit ball in $K\langle \pi T \rangle$ is the subspace of power series $\sum a_n T^n$, with $a_n \in o_K$, and whose coefficients satisfy

$$\omega(a_n) \geq n \text{ for all } n \geq 0.$$

One can verify that the norm topology on $K\langle T \rangle$ identifies this subset with the space of sequences $(\pi^n a_n)$ with $a_n \in o_K$, equipped with its product topology. As a product of compact sets this space is clearly compact by Tychonoff's theorem.

(See [**34**], the example following Remark 12.8). For a proof of compactness in our more general situation, see [**38**], Proposition 4.

A Fréchet space is a locally convex topological vector space that is complete and metrizable. The topology on a Fréchet space can be given by a countable family of semi-norms. Such spaces arise naturally as projective limits of Banach spaces – see [**34**, Chapter I, Section 8] for a general discussion.

In functional analysis, one can equip the space of continuous linear forms on a locally convex topological vector space with many different topologies. One of the most important of these is the "strong topology," which is the topology of uniform convergence on bounded subsets. If V is a topological vector space, we let V' be the vector space of continuous linear forms and V'_b be this space equipped with the strong topology (the "b" is for "bounded"). Recall that a topological vector space is reflexive if V is isomorphic to $(V'_b)'_b$ by the natural map from V to V'' given by evaluation. All of these topics are thoroughly treated in [**34**].

The principal consequence of this compactness is the following result describing $\mathcal{O}_{\mathcal{X}}$ as a topological vector space.

Proposition 1.2.6. *$\mathcal{O}_{\mathcal{X}}$ is a reflexive Fréchet space. The topology comes from the family of norms on the Banach algebras $\mathcal{O}(\mathcal{X}_n^-)$.*

For a proof, see [**34**], Proposition 16.5.

We next briefly recall the definition of a Stein space, following Kiehl ([**22**]):

Definition 1.2.7. A rigid space X is called a (quasi)-Stein space if there is an increasing sequence $U_1 \subset U_2 \subset \cdots$ of open affinoid subdomains of X forming an admissible covering such that the transition maps $\mathcal{O}(U_i) \to \mathcal{O}(U_{i-1})$ have dense image.

One can see that this density property holds for the transition maps $\mathcal{O}(\mathcal{X}_n^-) \to \mathcal{O}(\mathcal{X}_{n-1}^-)$ for $n \geq 2$ by considering the set of algebraic rational functions on $\mathbf{P}^1$ whose polar divisors are supported on the K-rational points. This set of rational functions forms a dense subring in each $\mathcal{O}(\mathcal{X}_n^-)$.

With regard to coherent sheaves, a Stein space behaves somewhat like an affine variety does in algebraic geometry. In particular, we have the following theorem (see [**22**]):

Proposition 1.2.8. *Let $\mathcal{M}$ be a coherent sheaf on $\mathcal{X}$. Then $H^i(\mathcal{X}, \mathcal{M}) = 0$ for $i > 0$ (Theorem B), and, if $M = H^0(\mathcal{X}, \mathcal{M})$ then the map $\mathcal{O}_{\mathcal{X}} \otimes M \to \mathcal{M}(\mathcal{X}_i^-)$ has dense image for any $i \geq 1$ (Theorem A).*

One can do even better. To give a coherent sheaf $\mathcal{M}$ on $\mathcal{X}$ is the same as giving, for each i, a finitely generated module M_i for $\mathcal{O}(\mathcal{X}_i^-)$. The global sections of this sheaf are an $\mathcal{O}_{\mathcal{X}}$-module

$$M = \operatorname{proj\,lim} M_i.$$

As is explained in [**16**] (see also [**42**]), one can recover the M_i as $\mathcal{O}(\mathcal{X}_i^-) \otimes M$. One can also characterize those M arising as global sections of a coherent sheaf by requiring that these tensor products be finitely generated.

1.3. The Reduction Map.

1.3.1. *The Bruhat-Tits Tree.* Our next task will be to introduce the Bruhat-Tits tree, which functions as a beautiful combinatorial approximation of $\mathfrak{X}$. We work always with the fixed two dimensional vector space V^* over K. By a lattice L in V^* we mean a free rank-two o_K module in V^*. We say two lattices L_1 and L_2 are equivalent if there is a scalar $a \in K$ so that $L_1 = aL_2$.

DEFINITION 1.3.1. Let X be the graph whose vertices are equivalence classes $[L]$ of lattices $L \subset V^*$, where two vertices x and y are joined by an edge if $x = [L_1]$ and $y = [L_2]$ with

$$\pi L_1 \subsetneq L_2 \subsetneq L_1.$$

PROPOSITION 1.3.2. *The graph X is a homogeneous tree of degree $q+1$.*

PROOF. The degree assertion follows from the fact that the edges leaving a given vertex $[L_1]$ correspond to the distinct lattices L_2 satisfying the adjacency relation

$$\pi L_1 \subsetneq L_2 \subsetneq L_1$$

and these in turn are in bijection with the one-dimensional $o_K/\pi o_K$-subspaces in the two-dimensional $o_K/\pi o_K$-vector space $L_1/\pi L_1$. There are $q+1$ such subspaces, so there are $q+1$ adjacent vertices.

Suppose that X is not a tree. A cycle in X would be minimally represented by a chain of lattices

$$L' \subsetneq L_d \subsetneq L_{d-1} \subsetneq \cdots L_1 \subsetneq L$$

where $L' = \pi^r L$ for some positive integer r and where none of the intermediate lattices are equivalent. Because L/L' is not a cyclic o_K-module, there is a smallest i such that L/L_i is cyclic but L/L_{i+1} is not. It follows that L_{i-1}/L_{i+1} is a non-cyclic, length 2 o_K-module, so $L_{i+1} = \pi L_{i-1}$. This contradicts the minimality of the representation of the cycle, and so we conclude that X has no cycles. $\square$

As constructed so far, the tree X is a combinatorial object. If we view each edge of X (with its bounding vertices) as a copy of the unit interval, we obtain a topological space called the geometric realization of X. Since it is this geometric realization that we are principally interested in, we will just go ahead and use the letter X to refer to it. A point on the edge in X joining the vertices $[L]$ and $[L']$ is determined by its barycentric coordinates: for $t \in [0,1]$, we write $x = (1-t)[L]+t[L']$ to indicate the point "at distance t from the vertex $[L]$ in the direction of $[L']$."

The group G acts on the lattices in V^* and on the tree X. The stabilizer of a lattice class $[L]$ is the subgroup of G generated by the center of G and by the compact open subgroup $\mathrm{GL}(L) \subset G$. If L is the lattice spanned by Ξ_0 and Ξ_1, then $\mathrm{GL}(L)$ is just $G_o = \mathrm{GL}_2(o_K) \subset G = \mathrm{GL}_2(K)$.

From now on, we let L_0 and L_1 be the lattices $\langle \Xi_0, \Xi_1 \rangle$ and $\langle \Xi_0, \pi\Xi_1 \rangle$ respectively. We will also write v_0 for the vertex of X corresponding to $[L_0]$, and e_0 for the edge of X running from v_0 to the vertex corresponding to L_1.

1.3.2. *Norms.* The tree X parameterizes norms on the two dimensional vector space V^* in a natural way. This description actually pre-dates Bruhat-Tits (see [**18**]).

DEFINITION 1.3.3. A norm on V^* is a function $\gamma : V^* \to \mathbf{R} \cup \{\infty\}$ such that

- $\gamma(x) = \infty$ if and only if $x = 0$.
- $\gamma(ax) = \omega(a) + \gamma(x)$ for $a \in K$.

- $\gamma(x+y) \geq \inf\{\gamma(x), \gamma(y)\}$

Two norms γ_1 and γ_2 on V^* are considered equivalent if $\gamma_1 - \gamma_2 = C$ for some constant $C \in \mathbf{R}$. Given a point $x \in X$, we may associate an equivalence class of norms on V^*. There are two cases to consider:

Case 1: x is a vertex. In this case, choose a lattice L representing x and let

$$\gamma(w) = -\inf\{n \in \mathbf{Z} : \pi^n w \in L\}.$$

Alternatively, choose a basis ℓ_0, ℓ_1 for L and define

$$\gamma(a\ell_0 + b\ell_1) = \inf\{\omega(a), \omega(b)\}.$$

Case 2: $x = (1-t)[L] + t[L']$. In this case, choose a basis ℓ_0, ℓ_1 for L such that L' is spanned by $\ell_0, \pi\ell_1$. Define

$$\gamma(a\ell_0 + b\ell_1) = \inf\{\omega(a), \omega(b) - t\}.$$

Notice that, in Case 2, the construction is consistent with Case 1 when $t = 0$ or $t = 1$. It's not hard to check, too, that Case 2 is compatible with different choices of lattices in the equivalence classes.

Proposition 1.3.4. *This construction establishes a bijection between the set of equivalence classes of norms on V^* and the points of the space X.*

Proof. We will construct an inverse map to the construction given above. Let γ be any norm on V^*. By translating γ in its equivalence class, we may assume that there is some $x \in V^*$ with $\gamma(x) = 0$. Let L' be the unit ball for γ:

$$L' = \{x \in V^* : \gamma(x) \geq 0\}.$$

Choose a (finite) set of representatives R in L for the projective space $\mathbf{P}(L'/\pi L')$. The norm γ is determined by its values on elements of R, all of which lie in $[0, 1)$. To see this, write any $w \in V^*$ as $w = u\pi^m r + \pi^{m+1} w'$ with $u \in o_K^*$ and $w' \in L'$. Then $\gamma(w) = m + \gamma(r)$. Now, if $\gamma(r) = 0$ for all $r \in R$, then γ is the norm associated to the lattice L as in Case 1 above. One can check further that, in this case, all norms equivalent to γ have unit balls equivalent to L', so the association of L' to γ makes sense. On the other hand, if there exists a point r with $\gamma(r) > 0$, then that r is unique. Indeed, if there were two such elements r and r', then these elements span L', from which it follows that $\gamma(x) > 0$ for all $x \in L'$, contrary to hypothesis. Now set $L = L' + r/\pi$. The norm γ, in this case, corresponds to the norm coming from Case 2 with the given lattice classes $[L]$ and $[L']$ and $t = 1 - \gamma(r)$. In this case, one checks that for norms equivalent to γ, the unit balls are either equivalent to L or to L'; and that if one follows the recipe given here to construct a point in X, one gets the same point regardless of which of these two possibilities holds for the chosen representative norm. □

1.3.3. *The group action.* The group action of G on X translates to the action on norms through the rule $(g \cdot \gamma)(x) = \gamma(g^{-1}x)$.

Lemma 1.3.5. *Some properties of the group action on X are:*

1. *The group G permutes the vertices and edges of X transitively.*
2. *The stabilizer of the lattice L_0 spanned by the standard coordinates Ξ_0 and Ξ_1, and the corresponding norm, is the subgroup $K^* \,\mathrm{GL}_2(o_K)$ in $\mathrm{GL}_2(K)$.*

(3) *If an element of G fixes the two endpoints of an edge, then it fixes the edge pointwise; the stabilizer of the edge in X corresponding to the lattice pair $\pi L_0 \subset L_1 \subset L_0$, where $L_1 = \langle \Xi_0, \pi\Xi_1 \rangle$ is, mod the center of G, the "Iwahori subgroup"*

$$B = \{g \in G_o : g \equiv \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \bmod \pi\}.$$

(4) *B is of index two in its normalizer; this normalizer is generated by B and any element of G that interchanges the two boundary vertices of the basic edge in (3). One such element is*

$$n = \begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}.$$

1.3.4. *Ends.* Let

$$([\Lambda_0], [\Lambda_1], \ldots)$$

be an infinite, non-backtracking sequences of adjacent vertices, which we can think of as an infinite ray in the tree heading off to ∞. Two such sequences are equivalent if they differ by a finite initial sequence of vertices, i.e.,

$$([\Lambda_0], [\Lambda_1], \ldots) \sim ([\Lambda'_0], [\Lambda'_1], \ldots)$$

if $[\Lambda_n] = [\Lambda'_{n+m}]$ for some fixed $m \in \mathbf{Z}$, and all n large enough. An equivalence class of such sequences is called an "end" of the tree. The set of ends of X is denoted $\mathrm{Ends}(X)$ and represents the set of points "at infinity" for the tree X.

To an oriented edge e running from $[\Lambda_0]$ to $[\Lambda_1]$, we associate the subset

$$U(e) = \{x \in \mathrm{Ends}(X) : x = ([\Lambda_0], [\Lambda_1], \ldots)\}.$$

The collection of sets $U(e)$, as e runs through the oriented edges of X, form the basis for a topology on $\mathrm{Ends}(X)$.

Given an end $x = ([\Lambda_0], [\Lambda_1], \ldots)$, we can construct a representing sequence of lattices

$$\Lambda_0 \supsetneq \Lambda_1 \supsetneq \Lambda_2 \supsetneq \cdots$$

with the property that Λ_i/Λ_{i+1} is isomorphic to $o_K/\pi o_K$. Since the sequence has no backtracking, the argument that we used in Proposition 1.3.2 tells us that Λ_0/Λ_i is a cyclic o_K-module of length i for each $i \geq 1$, and the same is true for $\Lambda_i/\pi^i\Lambda_0$. As a result we may choose $\ell_i \in \Lambda_0 \backslash \pi\Lambda_0$ such that

$$\Lambda_i = o_K \ell_i + \pi^i \Lambda_0.$$

Similarly,

$$\Lambda_{i+1} = o_K \ell_{i+1} + \pi^{i+1} \Lambda_0.$$

Because $\Lambda_{i+1} \subsetneq \Lambda_i$ and both ℓ_i and ℓ_{i+1} belong to $\Lambda_0 \backslash \pi\Lambda_0$, we must have

$$\ell_{i+1} \equiv a\ell_i \pmod{\pi^i \Lambda_0},$$

for some $a \in o_K^*$. We conclude that we may choose the ℓ_i to form a coherent sequence converging to a nonzero element ℓ of the intersection $\cap\Lambda_i$, and that this intersection is one-dimensional. The kernel of ℓ is a point of $\mathbf{P}^1$, denoted $N(x)$.

LEMMA 1.3.6. *The map $N : \mathrm{Ends}(X) \to \mathbf{P}^1$ is a G-equivariant homeomorphism.*

PROOF. Let $L_0 = o_K\Xi_0 + o_K\Xi_1$ as above. Given a point $[x, y]$ in $\mathbf{P}^1$, written with unimodular coordinates, let $\ell = -y\Xi_0 + x\Xi_1 \in L_0$. The end

$$(L_0, o_K\ell + \pi L_0, o_K\ell + \pi^2 L_0, \ldots)$$

maps, under N, to the point $[x, y]$. This shows the map is surjective. Conversely, we showed above that, if ℓ is a generator for the intersection of the sequence of lattices Λ_i representing an end

$$([\Lambda_0], [\Lambda_1], [\Lambda_2], \ldots).$$

with $\Lambda_0 = L_0$, then we must have

$$\Lambda_i = o_K\ell + \pi^i L_0,$$

and so the map N is bijective.

To complete the proof, observe that the image under N of the open set $U(e_0)$ determined by the edge e_0 is the set of points (unimodular as always) $[x, y]$ such that $ax+by = 0$ for some $a, b \in o_K$ with $a\Xi_0+b\Xi_1 \equiv \Xi_0 \pmod{\pi o_K}$. This is precisely the open set $\{[x, 1] : x \in \pi o_K\} \subset \mathbf{P}^1$. The G-equivariance of the map can be checked from the definitions, and using G-equivariance, one may conclude that N is open and continuous; since it is bijective we conclude that it is a homeomorphism. $\square$

1.3.5. *Group action on the ends.* The group G acts transitively on the ends. Furthermore:

(1) The stabilizer of an end is a Borel subgroup in G. In particular, the stabilizer of the end

$$([L_0], [\Xi_0 + \pi L_0], [\Xi_0 + \pi^2 L_0], \ldots)$$

is the subgroup

$$P = \{g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : g \in G\}.$$

(2) By (1), we may identify the ends (or $\mathbf{P}^1$) with G/P. To be completely explicit, the point $[x, y] \in \mathbf{P}^1$ corresponds to the coset

$$\begin{pmatrix} y & * \\ -x & * \end{pmatrix} P \in G/P.$$

In this identification, the open set corresponding to the edge e_0 is $BP/P \subset G/P$.

1.3.6. *The reduction map.* Given a point $x \in \mathcal{X}(\mathbf{C}_p)$ represented by homogenous coordinates $[a, b]$, we obtain a norm γ_x (defined up to equivalence) on V^* by setting

$$\gamma_x(\ell) = \omega(\ell(a, b))$$

for ℓ a linear form in V^*. The fact that this is indeed a norm is just a restatement of the fact that the coordinates a and b are linearly independent over K, which holds because the point x belongs to

$$\mathcal{X}(\mathbf{C}_p) = \mathbf{P}^1(\mathbf{C}_p)\backslash\mathbf{P}^1(K).$$

The map $x \mapsto [\gamma_x]$ from $\mathcal{X}(\mathbf{C}_p)$ to X is called the reduction map:

$$r : \mathcal{X} \to X$$

LEMMA 1.3.7. *The reduction map is G-equivariant, so $g(\gamma_x)(\ell) = \gamma_{gx}(\ell)$. Let $[L_0]$ be, as usual, the lattice spanned over o_K by Ξ_0 and Ξ_1, and L_1 the sublattice spanned by Ξ_0 and $\pi\Xi_1$. Then the inverse image under the reduction map of the vertex $[L_0]$ in $\mathcal{X}$ is the affinoid subdomain*

$$r^{-1}([L_0]) = \{[x,1] : x \in \mathbf{C}_p \text{ and } \omega(x-t) = 0 \text{ for all } t \in o_K\}$$

The inverse image of the open edge e_0 running from $[L_0]$ to $[L_1]$ is the admissible annulus

$$r^{-1}(e) = \{[x,1] : x \in \mathbf{C}_p \text{ and } 1 > \omega(x) > 0\}.$$

PROOF. The G-equivariance is a simple calculation. Let us therefore analyze the fibers of the reduction map. Consider first $r^{-1}(L_0)$. By definition, this is the set of points $[x, y]$ in unimodular coordinates such that

$$\omega(ax+by) = \inf\{\omega(a), \omega(b)\} \tag{1.3.8}$$

for all a and b in o_K. This equation is a fancy way to write the requirement that, writing $z = x/y$, we must have $\omega(az + b) = 0$ as a and b run through o_K. This is precisely the condition defining our affinoid in the lemma. For the inverse image of the edge, the condition we seek is

$$\omega(ax+by) = \inf\{\omega(a)+t, \omega(b)\}$$

for some real $1 > t > 0$. We may conclude that y is a unit (which might as well be 1) and that $\omega(x) = t$. Conversely, if $y = 1$ and $\omega(x) = t$, then we obtain the desired norm. Letting t vary between 0 and 1 gives us the full result. □

The affinoids $\mathcal{X}_n^-$ that we constructed to form an admissible covering of $\mathcal{X}$ are the inverse images under reduction of the subtrees of X made up of vertices and edges at distance at most $n-1$ from the fixed central vertex v_0.

It's also worth observing that points $[x, y] \in \mathbf{P}^1(K)$ give rise to seminorms on V^*, and that the kernel of such a seminorm corresponds to an end of X. One can extend the reduction map from $\mathcal{X}$ to all of $\mathbf{P}^1$, with the boundary points mapping to the ends – all in a G-equivariant way. For this approach to the higher dimensional building, see [**47**].

1.3.7. *The holomorphic discrete series.* We conclude this lecture by introducing certain spaces of functions on $\mathcal{X}$ that are closely related to modular forms. Let k be an even integer. Define $\mathcal{O}(k)$ to be the ring of entire functions on $\mathcal{X}$, equipped with the G-action:

$$g_*f = \frac{\det(g)^{k/2}}{(bz+d)^k} f\left(\frac{az+c}{bz+d}\right).$$

These spaces are called "the holomorphic discrete series" for $\mathrm{GL}_2(K)$. For their general construction, see [**32**].

In the important special case $k = 2$, we have a G-isomorphism

$$\begin{array}{rcl} \mathcal{O}(2) & \to & \Omega^1_{\mathcal{X}} \\ f & \mapsto & f\,dz. \end{array}$$

where we write $\Omega^1_{\mathcal{X}}$ for the global sections of the sheaf of one-forms on $\mathcal{X}$. More generally, for $k > 0$ and even we can identify $\mathcal{O}(k)$ with $(\Omega^1_{\mathcal{X}})^{\otimes k/2}$.

LEMMA 1.3.9. *The $(k-1)$-fold derivative map gives a G-equivariant map*

$$\begin{array}{rcl} \mathcal{O}(2-k) & \to & \mathcal{O}(k) \\ f & \mapsto & \left(\frac{d}{dz}\right)^{k-1} f \end{array}$$

The kernel of this map is the (finite dimensional) space of polynomials in z of degree at most $k-1$.

DEFINITION 1.3.10. We let $H_{DR}(k)$ be the cokernel of the derivative map defined above.

When $k = 2$, the Stein property of $\mathcal{X}$ implies that $H_{DR}(2) = H^1_{DR}(\mathcal{X})$.

2. Boundary distributions and integrals

2.1. Locally Analytic Functions and Distributions. The space of rigid analytic functions on $\mathcal{X}$ is isomorphic, via an integral transform, to a space of distributions on the boundary $\mathbf{P}^1$ of $\mathcal{X}$. This result, due originally to Morita, is a kind of p-adic analogue of the Poisson kernel from classical complex analysis. In order to introduce the space of distributions that concerns us, we need a brief digression on locally analytic functions.

2.1.1. *Locally analytic functions.* We first define locally analytic functions and manifolds. Given $\mathbf{a} = (a_1, \ldots, a_n) \in K^n$ and $\mathbf{r} \in \mathbf{R}^n$, let $B(\mathbf{a}, \mathbf{r})$ be the closed polydisc:

$$B(\mathbf{a}, \mathbf{r}) = \{(x_i)_{i=1}^n \in K^n : \omega(x_i - a_i) \geq r_i \text{ for } i = 1, \ldots, n\}.$$

A K-analytic function on such a disc (with values in a complete field L containing K) is given by a convergent power series

$$f(\mathbf{x}) = \sum_I c_I (\mathbf{x} - \mathbf{a})^I$$

Here the sum is over n-tuples $I = (i_1, \ldots, i_n)$ with $i_j \geq 0$, the coefficients $c_I \in L$, and

$$(\mathbf{x} - \mathbf{a})^I = \prod_{j=1}^n (x_j - a_j)^{i_j}.$$

The convergence condition is

$$\omega(c_I) + \sum_{j=1}^n r_j i_j \to \infty \text{ as } |I| \to \infty$$

where $|I| = \sum_{j=0}^n i_j$. Let us call the space of such analytic functions $\mathcal{A}_L(B(\mathbf{a}, \mathbf{r}))$. It is a Banach space with respect to the norm

$$\omega(f) = \inf_I \left\{ \omega(c_I) + \sum_{j=1}^n r_j i_j \right\}.$$

More generally, a K-analytic map between such discs is a map given by a collection of power series of this form.

DEFINITION 2.1.1. Let M be a paracompact topological space.

(1) A K-analytic chart (M_i, ϕ_i) for M is an open set M_i together with a homeomorphism

$$\phi_i : M_i \to B_i = B(0, \mathbf{r}) \subset K^d$$

for some radius $\mathbf{r}$.

(2) Two charts are compatible if the map

$$\phi_i \circ \phi_j^{-1} : B_j \to B_i$$

is given by an analytic function.

(3) A collection of compatible charts is called an atlas for M.

(4) M, together with a maximal atlas, is called a locally K-analytic manifold.

The atlas on M allows us to identify the analytic functions on the set M_i with those on the ball B_i via ϕ_i. We will write $\mathcal{A}(M_i, \phi_i)$ for this space of power series. One can show that any covering of a K-analytic manifold can be refined to a pairwise disjoint covering.

We will be interested in the following spaces viewed as K-analytic manifolds:

(1) The group $G = \mathrm{GL}_2(K)$ and the various subgroups G_o, B, and P introduced so far;

(2) The projective space $\mathbf{P}^1$.

Note the difference between $\mathbf{P}^1$ viewed as a rigid analytic space and as a K-analytic manifold!

Definition 2.1.2. Let M be a K-analytic manifold. The locally analytic functions on M (with values in a field L) are defined as follows. To each covering of M by disjoint charts (M_i, ϕ_i) we associate the space of functions

$$C^{an}(\{M_i, \phi_i\}) = \prod_i \mathcal{A}_L(M_i, \phi_i)$$

with its product topology. We define

$$C^{an}(M, L) = \varinjlim C^{an}(\{M_i, \phi_i\})$$

where the limit is over finer and finer coverings. We equip this space with the direct limit topology.

Two other important function spaces associated with a K-analytic manifold are the locally constant (or *smooth*) functions $C^\infty(M, K)$ and the continuous functions $C(M, K)$. We have

$$C^\infty(M, L) \subset C^{an}(M, L) \subset C(M, L).$$

The locally constant functions are closed in $C^{an}(M, L)$, and the analytic functions are dense in $C(M, L)$.

When M is compact, the coverings used to construct $C^{an}(M, L)$ are finite, and the spaces

$$\prod \mathcal{A}_L(M_i, \phi_i)$$

are Banach spaces. By the same reasoning that we sketched in 1.2.2, the transition maps in this direct limit are compact. A topological vector space that is the direct limit of a sequence of Banach spaces with compact transition maps is called a vector space *of compact type.*

With these preliminary remarks, the following results hold for spaces of compact type and their duals:

(1) Compact type spaces are reflexive and complete.
(2) Closed subspaces of compact type spaces are of compact type.
(3) The quotient V/U of a vector space of compact type by a closed subspace is of compact type.
(4) If $V = \varinjlim V_i$, with the V_i Banach spaces and the transition maps compact, then the strong dual V_b' is a Fréchet space and satisfies $V_b' = \varprojlim (V_i)_b'$.

See [**34**] Section 16 for points (1) and (4). For point (2), see [**23**] Theorem 7' and 8.

PROPOSITION 2.1.3. *Suppose that M is compact and L is locally compact. Then $C^{an}(M, L)$ is a vector space of compact type, hence reflexive and complete.*

The space $D(M, L)$ of analytic distributions on M is, by definition, the strong dual $C^{an}(M, L)_b'$ of the analytic functions on M.

2.1.2. *Locally analytic principal series representations.* Of particular interest to us in these lectures are the following spaces of locally analytic functions. For each even integer k, let χ_k be the character of the Borel subgroup P defined by the formula

$$\chi_k\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) = (a/d)^{(k/2)}.$$

The "locally analytic induction" $\mathrm{ind}_P^G(\chi_k)$ is the space

$$\mathrm{ind}_P^G(\chi_k) = \{f \in C^{an}(G, K) : f(gp) = \chi_k^{-1}(p)f(g) \text{ for } p \in P \text{ and } g \in G\}.$$

This space carries a G-action on the left by the rule $g(f)(h) = f(g^{-1}h)$. It is an example of a locally analytic G-representation; for more on such representations, see [**39**] and [**42**].

We will interpret the representation $\mathrm{ind}_P^G(\chi_k)$ as the space of "locally meromorphic functions on $\mathbf{P}^1$ having poles only at infinity of order at most $-k$." To be a bit clearer about what we mean, recall that, in Section 1.3.5, we pointed out that $\mathbf{P}^1 \xrightarrow{\sim} G/P$ via the identification

$$[a, b] \mapsto \begin{pmatrix} b & * \\ -a & * \end{pmatrix} P.$$

At the cost of singling out the point $[1, 0]$ as the point at infinity, we can pullback a function $f \in \mathrm{ind}_P^G(\chi_k)$ to K via the map

$$x \mapsto u(x) = \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix}.$$

This function is locally analytic on K. However, it enjoys the stronger properties that there is an integer N such that

- f is locally analytic on the set of $x \in K$ with $\omega(x) \geq N$, and
- we have a convergent expansion

$$f(z) = \sum_{i \geq k} c_i z^{-i}$$

on the set where $\omega(z) < N$.

The group action on $\mathrm{ind}_P^G(\chi_k)$ becomes the action

$$g(f) = \frac{(ad - bc)^{k/2}}{(bz + d)^k} f\left(\frac{az + c}{bz + d}\right),$$

which clearly preserves the space of functions satisfying the conditions above. This is the space of functions we say are locally analytic, except have a pole of order at most $-k$ at infinity.

DEFINITION 2.1.4. For $k \leq 0$, Let $C^{an}(K,k)$ be the space of "locally analytic functions on K with at most a pole of order $-k$ at infinity", as defined above, with its associated group action. The topology on $C^{an}(K,k)$ can be defined as follows. Choose an integer N and a finite set $S_N = \{a_j\}$ of elements in K, so that one can cover K by the finite collection of balls

$$D(a_j, N) := \{x \in K : \omega(x - a_j) \geq N\},$$

together with

$$D(\infty, N) := \{x \in K : \omega(x) \leq N\}.$$

The analytic functions $\mathcal{A}(D(a_j,N))$ on $D(a_j,N)$ are given by convergent power series

$$f(x) = \sum_{i=0}^{\infty} c_i (x - a_j)^i$$

where $\omega(c_i) + iN \to \infty$ as $i \to \infty$. $\mathcal{A}(D(a_j,N))$ is a Banach space for the norm

$$\omega(f) = \inf\{\omega(c_i) + iN\}.$$

The "analytic functions with poles at infinity" $\mathcal{A}(D(\infty,N))$ on $D(\infty,N)$ are given by

$$f(x) = \sum_{i=k}^{\infty} c_i x^{-i}$$

where $\omega(c_i) + iN \to \infty$ as $i \to \infty$, with the norm

$$\omega(f) = \inf\{\omega(c_i) + iN\}.$$

The topology on $C^{an}(K,k)$ is the direct limit topology:

$$C^{an}(K,k) := \varinjlim_{N \to \infty} \prod_{a \in S_N \cup \{\infty\}} \mathcal{A}(a,N).$$

If $k \leq 0$ and even, and we look at $\mathrm{ind}_P^G(\chi_k)$ as $C^{an}(K,k)$, then we can identify two G-invariant subspaces inside it. The first one is the finite dimensional space

$$P_{-k} := \text{the space of polynomials of degree at most } -k.$$

The second is the space of functions $f(z)$ that are "locally polynomial functions" on K of degree at most $-k$, meaning that, for some covering of K by $D(a,N)$ (including $D(\infty,N)$), the restriction of f to each disc is a polynomial of degree at most $-k$. We let $C^{la}(K,k)$ denote this space. (The la stands for "locally algebraic.")

For the sake of concreteness it is also worth noticing that, even though elements of $C^{an}(K,k)$ are not strictly speaking functions on $\mathbf{P}^1$ (because they have poles at ∞), every element of the quotient space $C^{an}(K,k)/P_{-k}$ can be represented by a unique, truly locally analytic function on $\mathbf{P}^1$ that vanishes at ∞ — just subtract off the polar part at infinity using P_{-k}.

The methods of the paper [**37**] prove that the following sequence is exact, and that the representation $C^{an}(K,k)$ has length 3:

$$0 \to C^{la}(K,k)/P_{-k} \to C^{an}(K,k)/P_{-k} \xrightarrow{(\frac{d}{dx})^{1-k}} C^{an}(K,2-k) \to 0. \tag{2.1.5}$$

The locally algebraic part of the representation $C^{an}(K,k)$ (for $k \leq 0$) can be decomposed as a tensor product of the smooth representation $C^\infty(\mathbf{P}^1,K)/K$ and the finite dimensional representation P_{-k}.

2.2. The Integral Transform and Morita Duality. The locally analytic representations $\mathrm{ind}_P^G(\chi_k)$ discussed in the previous section are closely related to the topological vector spaces $\mathcal{O}(k)$ that come from functions on the p-adic upper half plane. One way to formulate this relationship, which we call Morita duality (see [**26**]), is by an integral transform.

Suppose that $\lambda : \mathcal{O}(k) \to K$ is a continuous linear functional. We may construct from λ a function $I_k(\lambda)$ on K via the formula

$$I(\lambda)(x) = \lambda\left(\frac{1}{z-x}\right).$$

Our goal in this section is to prove the following theorem.

THEOREM 2.2.1. *For $k \geq 2$ and even, the map I_k yields a topological isomorphism*

$$\mathcal{O}(k)'_b \to C^{an}(K, 2-k)/P_{k-2}.$$

To prove the theorem, we will proceed in stages. First, we check the G-action. Substituting in the various definitions, we obtain:

$$g(I_k(g^{-1}(\lambda)))(x) = \frac{(bx+d)^{k-2}}{(ad-bc)^{k/2-1}}\lambda\left(\frac{(ad-bc)^{k/2-1}}{(bz+d)^k}\frac{(bz+d)(bx+d)}{z-x}\right).$$

Now using the fact that

$$(bz+d)(bx+d) = (bz+d)^2 - (z-x)b(bz+d)$$

one obtains

$$g(I_k(g^{-1}(\lambda)))(x) = \lambda\left(\left(\frac{bx+d}{bz+d}\right)^{k-2}\left(\frac{1}{z-x} + c(z)\right)\right)$$

where $c(z)$ is independent of x. Finally,

$$\left(\frac{bx+d}{bz+d}\right)^{k-2} = 1 + (z-x)H,$$

where H is a polynomial in x of degree $k-2$, with coefficients rational functions in z. Thus

$$g(I_k(g^{-1}(\lambda)))(x) \equiv \lambda(x) \pmod{P_{k-2}},$$

proving G-equivariance (formally).

Next, we prove that the function $I_k(\lambda)$ belongs to $C^{an}(K, 2-k)$. Functional analysis tells us that any continuous linear form λ on $\mathcal{O}(k)$ is induced by a continuous linear form $\lambda_n : \mathcal{O}(\mathcal{X}_n^-) \to K$ for some integer n. More precisely, we have the following relationship between these spaces and their strong duals (see [**34**], Proposition 16.5; the subscript b refers to the strong topology):

$$(\mathcal{O}_{\mathcal{X}})'_b = (\varprojlim \mathcal{O}(\mathcal{X}_n^-))'_b \xrightarrow{\sim} \varinjlim(\mathcal{O}(\mathcal{X}_n^-))'_b. \tag{2.2.2}$$

Let us choose representatives $\{a_i\}$ for $o_K/\pi^n o_K$ and $\{b_j\}$ for $\pi o_K/\pi^n o_K$ as in Setion 1.2.1, with $b_0 = 0$. The balls $D(a_i, n)$, $D(1/b_j, n-2\omega(b_j))$, and $D(\infty, n)$

form a covering of K as in the discussion following Definition 2.1.4. Suppose that $x \in D(a_i, n)$. Then

$$\frac{1}{z-x} = \frac{1}{(z-a_i)-(x-a_i)} = \sum_{j=0}^{\infty} \frac{(x-a_i)^j}{(z-a_i)^{j+1}},$$

the geometric series converging when $z \in \mathcal{X}_n^-$. The continuity of λ on $\mathcal{O}(\mathcal{X}_n^-)$ means that

$$\omega(\lambda(z-a_i)^{-j-1}) \geq C - (n-1)(j+1)$$

for some constant C. For $x \in D(a_i, n)$, we have

$$\lambda\left(\frac{1}{z-x}\right) = \sum_{j=0}^{\infty} \lambda\left(\frac{1}{(z-a_i)^{j+1}}\right)(x-a_i)^j,$$

and the series on the right converges because

$$C-(n-1)(j+1)+nj = C+j+1-n \to \infty \text{ as } j \to \infty.$$

This exhibits $I_k(\lambda)$ as an analytic function on $D(a_i, n)$. On $D(\infty, n)$, we see that

$$\lambda\left(\frac{1}{z-x}\right) = \lambda\left(\frac{1}{x(z/x-1)}\right) = -\sum_{j=0}^{\infty} \lambda(z^j)x^{-j-1}$$

and the inequalities bounding λ, defining $D(\infty, n)$ and $\mathcal{O}(\mathcal{X}_n^-)$ guarantee convergence as in the case considered earlier.

Notice that this calculation for $I_k(\lambda)$ on $D(\infty, n)$ actually proves more — the function $I_k(\lambda)(x)$ vanishes at the point ∞. This will enable us to settle the next step in our proof, which is to show that $I_k(\lambda)$ is injective. Because $I_k(\lambda)$ vanishes at infinity, it belongs to P_{k-2} only if it is identically zero. From the computations above, we see that $I_k(\lambda) = 0$ if and only if λ vanishes on the functions z^j and $1/(z-a)^j$ for all non-negative integers j and all $a \in K$. This in turn implies that λ vanishes on the rational functions in z having poles at rational points of $\mathbf{P}^1(K)$. Since these rational functions are dense in each $\mathcal{O}(\mathcal{X}_n^-)$, and λ is continuous, it follows that λ must be identically zero.

In light of the functional-analytic fact (2.2.2), the continuity of I_k is implicit in the calculation above. More precisely, we showed that I_k, restricted to those λ which factor through $\mathcal{O}(\mathcal{X}_n^-)$, is a bounded linear functional from this Banach space to the Banach space of locally analytic functions that are analytic for the specific covering we used in the calculation. This implies that I_k is continuous.

We must show that I_k is surjective, and that it is a topological isomorphism. In fact, the second claim follows from the first by the open mapping theorem (see [**34**], Proposition 8.8). To prove surjectivity, we will construct linear forms λ of a particular form. This will require something of a digression.

2.2.1. *Residues.* Given an edge e of the tree X, we know that the fiber $r^{-1}(e)$ of the reduction map at e is an admissible open set that is an annulus. Let us look for the moment at the particular admissible open set

$$U = r^{-1}(e_0) = \{[a,1] : 1 > \omega(a) > 0\} \subset \mathcal{X}$$

described in Lemma 1.3.7. This space is a union of affinoid subdomains

$$U_n = \{[a,1] : 1-(1/n) \geq \omega(a) \geq (1/n)\}$$

as $n \to \infty$, and the space of rigid functions on U is the Fréchet space arising as the projective limit of the corresponding affinoid algebras:

$$\mathcal{O}(U) = \varprojlim \mathcal{O}(U_n).$$

In concrete terms, $\mathcal{O}(U)$ consists of power series

$$f(z) = \sum_{z \in \mathbf{Z}} c_j z^j$$

that converge on each U_n. This condition amounts to the requirement that

$$\omega(c_j) + j/n \to \infty$$

for all n and $j \to \infty$, and

$$\omega(c_j) + j(1 - 1/n) \to \infty$$

as for all n as $j \to -\infty$. The family ρ_n of seminorms defining the topology are

$$\rho_n(f) = \inf\{\omega(c_j) + j/n, \omega(c_j) + j(1 - 1/n)\}.$$

Proposition 2.2.3. *The (rigid) DeRham cohomology of the annulus U is one dimensional and spanned by dz/z.*

Proof. The only obstacle to formal integration of a rigid function on U to obtain another rigid function is dz/z. □

Let Res be the isomorphism $\mathrm{Res} : H^1_{DR}(U) \to K$ such that $\mathrm{Res}(dz/z) = 1$.

Definition 2.2.4. Given an oriented edge e of X and a rigid analytic one-form $f(z)dz$ in $\Omega^1(\mathcal{X})$, we define $\mathrm{Res}_e(fdz)$ to be $\mathrm{Res}(g^{-1}(fdz)|U)$ where $g \in G$ is any element such that $ge_0 = e$.

This definition makes sense because the Iwahori group B acts trivially on $H^1_{DR}(U)$ and preserves the orientation of e_0. The normalizer of B, which reverses the edge e_0, sends z to π/z. It follows that $\mathrm{Res}_{e'}(fdz) = -\mathrm{Res}_e(fdz)$ if e and e' are opposite to one another.

2.2.2. *Surjectivity of the integral transform.* We can now use the residue map to prove that our integral transform I_k is surjective. Because of the G-equivariance of I_k, it suffices to prove that any analytic function on $o_K \subset K$ is in the image of I_k. Let

$$f(x) = \sum_{j=0}^{\infty} b_j x^j,$$

where $\omega(b_j) \to \infty$ as $j \to \infty$, be the desired target function. We will find a linear form λ such that $I_k(\lambda) = f$. To do this, let f_a be the restriction of f to the disk $D(a, 1) = a + \pi o_K$, where a runs through a set of representatives for $o_K/\pi o_K$. The function f_0 is given by the same power series as our original function f. The other functions f_a can be written as power series

$$f_a(z) = \sum_{j=0}^{\infty} b_j^a (x - a)^j$$

where the coefficients b_j^a satisfy $\omega(b_j^a) \to \infty$ as $j \to \infty$. Note that these representative power series on the disks $D(a, 1)$ are overconvergent — they converge on the bigger disk $D(0,0)$. Each f_a is a translate, under the group action, of an overconvergent power series on $D(0,1)$ like f_0. Thus, to prove surjectivity, it suffices to prove that f_0 is in the image of I_k. For this, we use the following lemma.

LEMMA 2.2.5. *Let $\{b_j\}_{j=0}^{\infty}$ be a sequence of elements of K with the property that, for some integer $n > 0$, $\omega(b_j) + j/n \to \infty$. Then*

$$f \mapsto \sum_{j=0}^{\infty} b_j \operatorname{Res}_e(z^j f)$$

is a continuous linear form on $\mathcal{O}(U)$.

PROOF. This follows from a computation with the semi-norms defining the topology on $\mathcal{O}(U)$. $\square$

The lemma applies to the particular coefficients of our analytic function f. We compute

$$I_k(\lambda)(x) = \sum_{j=0}^{\infty} b_j \operatorname{Res}\left(\frac{z^j}{z-x}\right).$$

Now we distinguish two cases. When $x \in D(0,1)$ and $z \in U$, we have $\omega(x) > \omega(z)$ and we see from the geometric series that

$$\frac{z^j}{z-x} = \sum_{\ell=0}^{\infty} z^{j-1-\ell} x^{\ell}.$$

As a result, we have

$$I_k(\lambda)(x) = \sum_{j=0}^{\infty} b_j x^j = f(x).$$

On the other hand, when $x \notin D(0,1)$, but $z \in U$, we have $\omega(x) < \omega(z)$ and thus we obtain the expansion

$$\frac{z^j}{z-x} = \sum_{\ell=0}^{\infty} z^{\ell+j} x^{-\ell-1}$$

and all residues of this function vanish. Therefore $I_k(\lambda)$ is supported on $D(0,1)$, where it agrees with $f(x)$. This proves surjectivity, and completes the proof of Theorem 2.2.1.

2.2.3. *The Poisson Kernel.* The Poisson Kernel J_k is the transpose of the map I_k:

$$J_k : (C^{an}(K, 2-k)/P_{k-2})'_b \to \mathcal{O}(k).$$

PROPOSITION 2.2.6. *Let μ be a continuous linear form on $C^{an}(K, 2-k)$ vanishing on P_{k-2}. The transpose J_k is given by*

$$J_k(\mu)(z) = \int_{\mathbf{P}^1(K)} \frac{1}{z-x}\, d\mu.$$

PROOF. Much of this calculation reproduces what we did in the proof of the main theorem. For example, the G-equivariance follows by essentially the same argument that we used earlier. We need to prove that $J_k(\mu)$ is rigid analytic on $\mathcal{X}$, and that $\lambda(J_k(\mu)) = \mu(I_k(\lambda))$. The second of these properties is formal once we know the analyticity, so we will focus on that. Choose a large integer N and representatives a_i for $o_K/\pi^N o_K$ and b_j for $\pi o_K/\pi^N o_K$ (with $b_0 = 0$) so that the balls $D(a_i, N)$, $D(1/b_j, N - 2\omega(b_j))$ for $j \neq 0$, and $D(\infty, N)$ cover K, as in (1). Then

$$J_k(\mu)(z) = \sum \int_D \frac{1}{z-x} d\mu$$

where the sum is over the discs in the covering. For a typical such disc $D(a, N)$, we have

$$\frac{1}{z-x} = \frac{1}{(z-a)-(x-a)} = \sum_{\ell=0}^{\infty} \frac{(x-a)^\ell}{(z-a)^{\ell+1}}$$

converging when $\omega(x-a) > \omega(z-a)$, in particular when $z \in \mathcal{X}_N^-$. The continuity of the distribution μ means that

$$\omega\left(\int_{D(a,N)} (x-a)^\ell d\mu\right) \geq C + N\ell$$

for some constant C. Applying this to the sum, we obtain

$$\int_{D(a,n)} \frac{1}{z-x} d\mu = \sum_{\ell=0}^{\infty} \frac{\int_{D(a,n)} (x-a)^\ell d\mu}{(z-a)^{\ell+1}}.$$

Since $\omega(z-a) \leq N-1$ on $\mathcal{X}_N^-$, we see that this series gives a rigid function on $\mathcal{X}_N^-$. Assembling the different discs shows that $J_k(\mu)$ is in fact rigid analytic on $\mathcal{X}_N^-$. □

COROLLARY 2.2.7. *Let $f \in \mathcal{O}(k)$ be a rigid function, and choose $N > 0$. Let a_i, b_j be chosen as in the proof of the theorem (or as in equation (1)). Then f restricted to $\mathcal{X}_N^-$ has a "partial fraction expansion"*

$$f(z) = \sum_{j=0}^{\infty} c_j^\infty z^j + \sum_{i=0}^{q^N-1} \sum_{\ell=1}^{\infty} \frac{c_\ell^i}{(z-a_i)^\ell} + \sum_{i=1}^{q^{N-1}-1} \sum_{\ell=1}^{\infty} \frac{d_\ell^i}{(z-1/b_i)^\ell}.$$

2.2.4. *Morita Duality.* We have shown that there is a duality pairing (first established by Morita ([**26**]) for $k \geq 2$:

$$\mathcal{O}(k) \times C^{an}(K, 2-k)/P_{k-2} \to K$$

given by

$$\langle F(z), f(x)\rangle = I_k^{-1}(f)(F) = J_k^{-1}(F)(f).$$

We can refine our understanding of this duality by looking more closely at the Jordan-Holder factors of the locally analytic representation on the right. Let us look at the restriction of the pairing to the subspace of locally polynomial functions:

$$\mathcal{O}(k) \to (C^{la}(K, 2-k)/P_{2-k})'_b.$$

Functional analysis tells us that this map is surjective. Furthermore, if we refer back to the definition of locally analytic functions on $\mathbf{P}^1(K)$, we see that the topology induced on the locally polynomial subspace $C^{la}(K, k)$ comes from viewing $C^{la}(K, k)$ as the direct limit of its finite dimensional subspaces. (This is because the set of locally polynomial functions relative to a fixed covering of $\mathbf{P}^1(K)$ is finite dimensional, and these finite dimensional subspaces are cofinal with all such subspaces.) The continuous dual, with respect to this topology, is just the full linear dual. The subspace $(C^{la}(K, k)/P_{k-2})'$ consists of all linear functionals on $C^{la}(K, k)$ that vanish on P_{k-2}. To make this map more explicit, let us extract the following piece of information from the proof of the main theorem.

LEMMA 2.2.8. *Let $P(z)$ be a polynomial of degree at most $k-2$, and let $\lambda(f) = \mathrm{Res}_e(P(z)f(z)dz)$. Then $I_k(\lambda)$ is the function in $C^{an}(K, 2-k)/P_{k-2}$ equal to $P(x)$ on $D(0,1) = \pi o_K$ and zero elsewhere.*

Now we can make the dual of the locally polynomial functions completely explicit. To give a linear form λ on $C^{la}(K, 2-k)/P_{k-2}$, it suffices to know the values $\lambda(P(x)|U(e))$ for all polynomials $P(x)$ of degree at most $k-2$ and all open sets $U(e)$ corresponding to edges e of X. We can collect this information in a function

$$c_\lambda : \mathrm{Edges}(X) \to \mathrm{Hom}(P_{k-2}, K)$$

defined by $c_\lambda(e) = (P(x) \mapsto \lambda(P(x)|U(e)))$.

To make sure that the linear form λ vanishes on P_{k-2}, we need two properties:

(1) $c_\lambda(e') = -c_\lambda(e)$, when e' is the edge obtained by reversing e.
(2) c_λ is harmonic, meaning

$$\sum_{e \mapsto v} c_\lambda(e) = 0$$

where the sum is over the edges leaving a given vertex.

DEFINITION 2.2.9. Let M be an abelian group. Then a function $c : \mathrm{Edges}(X) \to M$ is called an (M-valued) harmonic cocycle if it satisfies the two conditions given above.

In our special case, given $F(z) \in \mathcal{O}(k)$ (with $k \geq 2$ as usual), we define a function

$$\begin{array}{rcl} c_F : \mathrm{Edges}(X) & \to & \mathrm{Hom}(P_{k-2}, K) \\ c_F(e)(P(x)) & = & \langle F, P(x)|U(e)\rangle. \end{array}$$

The function c_F is determined by the residue map – indeed, suppose that $e' = ge$, where e is the original basic edge used to define Res_e. Then

$$\begin{array}{rcl} c_F(e')(x^j) & = & \langle F, x^j|U(e')\rangle = \langle F, g((g^{-1}(x^j)|U(e))\rangle \\ & = & \langle g^{-1}(F), g^{-1}(x^j)|U(e)\rangle. \end{array}$$

Substituting in the definitions of the group actions, and remembering that the space of polynomials P_{k-2} is a subset of the space of locally algebraic functions $C^{la}(K, 2-k)$ we see that

$$c_F(e') = \mathrm{Res}_e(g_*^{-1}(z^j F(z)dz)) \tag{2.2.10}$$

where the group action is the usual action on differentials — ignoring k. For this reason, we call the map $F \mapsto c_F$ the residue map.

DEFINITION 2.2.11. Let $C_{har}(k)$ be the space of harmonic functions on the edges of the tree X with values in $\mathrm{Hom}(P_{k-2}, K)$.

Referring back to Equation 2.1.5, we have the following commutative diagram for $k \geq 2$:

$$\begin{array}{ccccc} (C^{an}(K, 2-k)/P_{k-2})'_b & \longrightarrow & (C^{la}(K, 2-k)/P_{k-2})'_b & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ \mathcal{O}_{\mathcal{X}}(k) & \xrightarrow{\mathrm{Res}} & C_{har}(k) & \longrightarrow & 0 \end{array}$$

The remaining question for understanding $\mathcal{O}_{\mathcal{X}}(k)$, for $k \geq 2$ is to understand the kernel of the residue map. This is answered by the following result

THEOREM 2.2.12. *The kernel of the residue map is the image of* $\mathcal{O}_{\mathcal{X}}(2-k)$ *in* $\mathcal{O}_{\mathcal{X}}(k)$ *(see Lemma 1.3.9). In particular:*

(1) *This image is closed;*
(2) $C_{har}(k) \xrightarrow{\sim} H_{DR}(k)$*;*
(3) $\mathcal{O}_{\mathcal{X}}(2-k)/P_{k-2} \xrightarrow{\sim} C^{an}(K,k)'_b$ *for* $k \geq 0$ *and even.*

PROOF. It is easy to see that the image of $\mathcal{O}_{\mathcal{X}}(2-k)$ in $\mathcal{O}_{\mathcal{X}}(k)$ belongs to the kernel of the residue map. Indeed, if

$$f = \sum_{i=-\infty}^{\infty} c_i z^i$$

then

$$\frac{d^j f}{dz^j} = \sum_{i=-\infty}^{-j-1} b_i z^i + \sum_{i=0}^{\infty} b_i z^i$$

for some constants b_i — from this it is clear that $\operatorname{Res}(P(z)f(z)dz) = 0$ for all polynomials $P(z)$ of degree at most $j-1$. Conversely, a function $g(z)$ on U is in the image of the $(1-k)^{th}$ derivative if and only if the coefficients c_i, for $i = -1, \ldots, k-1$, are all zero. Since that image is G-equivariant, we can conclude that $\operatorname{Res}_e(P(x)F(x)dx) = 0$ for all e if F is in the image. Working with the derivatives, one can further check that the following diagram commutes:

$$\begin{array}{ccccc} C^{an}(K,2-k) & \longrightarrow & C^{an}(K,k) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ \mathcal{O}_{\mathcal{X}}(k)'_b & \longrightarrow & \mathcal{O}_{\mathcal{X}}(2-k)'_b & & \end{array}$$

where the upper arrow comes from Equation 2.1.5 and the lower one is the dual map to the derivative map in Lemma 1.3.9. Therefore, the key point is that $\mathcal{O}_{\mathcal{X}}(2-k)$ has closed image. We won't give all the details of the proof of this, but it follows from the "partial fractions" decomposition given in Corollary 2.2.7. The idea is to see that, if $\operatorname{Res}(z^j f(z))$ vanishes on all edges e, then the terms of the form $1/(z-a)^j$ in the partial fractions decomposition vanish for $0 \leq j \leq k-2$. Then one can formally integrate the partial fractions decomposition $k-1$ times and obtain a rigid function that still converges on some $\mathcal{X}_n^-$. These integrals can be glued together because the obstruction to doing so lies in $H^1(\mathcal{X}, \mathcal{O})$, which is zero by the Stein property. $\square$

2.3. Bounded Distributions. The bounded harmonic functions, relative to a suitably chosen norm, play a special role in the analytic theory of the p-adic upper half plane. To explore this, choose a norm on P_{k-2} (for $k \geq 2$) that is invariant by the Iwahori group B which stabilizes our standard edge e_0. (There are many such choices; for example, the sup-norm on the coefficients of the polynomials in P_{k-2} will do). We will use the same notation ω for this norm, and for the associated dual norm on $\operatorname{Hom}(P_{k-2}, K)$.

If $c \in C_{har}(k)$, we say that c is *bounded* if

$$\omega(c) = \inf_{g \in G/B} \omega(g^{-1}(c(ge_0))) \tag{2.3.1}$$

exists. Notice that $\omega(c)$ is well-defined, because the B-invariance of the norm on $\mathrm{Hom}(P_{k-2}, K)$ means that the terms in the infimum are independent of the choice of coset representatives. We write $C^b_{har}(k)$ for the space of bounded harmonic functions — they form a Banach space with respect to the given G-invariant norm.

The bounded elements in $C_{har}(2)$ are the harmonic functions whose values are p-adically bounded.

The boundedness condition translates into an estimate for the "integrals" of locally polynomial functions in $C^{la}(K, 2-k)$. This, in turn, leads to the following version of the "Theorem of Amice-Velu-Vishik."

THEOREM 2.3.2. *Suppose that c is a bounded harmonic function in $C_{har}(k)$. Then there is a unique continuous linear form $\lambda_c : C^{an}(K, 2-k) \to K$ that vanishes on P_{k-2} and satisfies the following conditions:*

(1) $\lambda_{gc} = g(\lambda_c)$.
(2) $\lambda_c(P(x)|U(e_0)) = c(e_0)(P(x))$ *for $P(x)$ of degree at most $k-2$.*
(3) *There is a constant A such that, for all $n \geq 0$, $m \geq 0$, and $a \in o_K$, we have*

$$\omega(\lambda_c((x-a)^n | a + \pi^m o_K)) \geq A + m(j - 1 - k/2).$$

(4) *We have*

$$\lambda_c\left(\left(\sum_{m=0}^{\infty} c_m (x-a)^m\right) | a + \pi^m o_K\right) = \sum_{m=0}^{\infty} c_m \lambda((x-a)^m | a + \pi^m o_K).$$

PROOF. We only sketch the proof. A computation with the various group actions shows that conditions (1) and (2) give us a well-defined way to compute $\lambda_c(P(x)|U)$ for any compact open set U in $\mathbf{P}^1$ and any polynomial $P \in P_{k-2}$. The harmonicity of c implies that λ_c vanishes on P_{k-2}. To integrate a locally polynomial function, choose $g \in G$ carrying U to the standard open set $U(e_0)$ and compute

$$\lambda_c(P(x)|U) = \lambda_c(P(x)|g^{-1}(U(e_0))) = \lambda_{gc}(g^{-1}(P)|U(e_0)).$$

(one checks that this does not depend on the choice of g.) The boundedness property of c turns into the estimate (3), at least for $0 \leq n \leq k-2$. If e is an edge such that $U(e)$ does not contain ∞, then it must be of the form $a + \pi^m o_K$. In that case, we wish to estimate, for $0 \leq n \leq k-2$, the value of λ_c:

$$\lambda_c((x-a)^n | a + \pi^m o_K)) = \lambda_c(g^{-1}([\pi^{(j-(k-2)/2)m} x]^n | o_K) \tag{2.3.3}$$

where

$$g = \begin{pmatrix} 1 & 0 \\ a\pi^{-m} & \pi^{-m} \end{pmatrix}.$$

Let A be $\omega(c) = \omega(gc)$. Then

$$\omega(g(\lambda_c)(x^n | o_K)) = c(e_0)(x^n) \geq A.$$

Combined with equation 2.3.3 we obtain (3).

Finally, we show how to compute $\lambda_c(f|U(e_0))$ for locally analytic f. Cover $U(e_0) = \pi o_K$ by open sets $a + \pi^m o_K$ for some large m. On each open set, let $P_{a,m}$ be the truncation of the Taylor expansion of f on the disc $a + \pi^m o_K$ obtained by discarding terms of degree greater than $k-2$. Define

$$S_m = \sum_a \lambda_c(P_{a,m} | a + \pi^m o_K)$$

using the fact that we know how to integrate polynomials of low degree. Then the estimate (3) implies that the limit, as $m \to \infty$, of S_m exists. This gives our integral. See [**25**, Section 11] for one proof with details. See [**6**, Theorem 2.5] for another proof. $\square$

COROLLARY 2.3.4. *Let $\mathcal{O}_{\mathcal{X}}(k)^b$ be the space of rigid functions F such that* $\mathrm{Res}(F)$ *is a bounded harmonic function. The residue map gives an isomorphism between $\mathcal{O}_{\mathcal{X}}(k)^b$ and $C_{har}(k)^b$; the inverse of this map is the Poisson integral.*

PROOF. The kernel function $\frac{1}{z-x}$ is locally analytic; given a bounded harmonic function, we can apply the corresponding linear form to it. The proof that the result is rigid analytic is another argument with the geometric series that relies on the estimate (3) to obtain convergence. $\square$

The bounded functions $\mathcal{O}_{\mathcal{X}}(k)$ can be characterized differently. The spaces $\mathcal{O}(\mathcal{X}_n^-)$ are Banach spaces; fix one such n and let $\omega(F)$ denote, for the moment, the norm of a function F restricted to $\mathcal{O}(\mathcal{X}_n^-)$.

THEOREM 2.3.5. *The residues* $\mathrm{Res}(F)$ *are bounded, and $F \in \mathcal{O}_{\mathcal{X}}(k)^b$ if and only if $\omega(gF) \geq C$ for some constant C and all $g \in G$.*

PROOF. See [**10**]. $\square$

2.4. Discrete groups, modular forms, and uniformization. The p-adic upper half plane was originally introduced by Mumford as a way to construct families of algebraic curves lying at the boundary of moduli space. Mumford showed that, for appropriate discrete subgroups $\Gamma \subset G$, the quotient $\mathcal{X}/\Gamma$ has the structure of an algebraic curve.

The work of Cerednik and Drinfeld made clear the arithmetic significance of Mumford's p-adic uniformization theory. They showed that one could construct Shimura curves — modular curves parameterizing abelian surfaces with quaternionic multiplication — via p-adic methods.

We will recall a few of the features of this theory. For more of the story, see the work of Gerritzen and van der Put ([**17**]) or Mumford's original paper ([**27**]). For the arithmetic theory and uniformization of Shimura curves, see Drinfeld's original (7 page) paper ([**12**]) or the book by Boutot and Carayol that explains that paper in detail ([**4**]).

Choose a definite quaternion algebra B over $\mathbf{Q}$ with discriminant N. From the theory of such algebras, we know that N must be a squarefree integer with an odd number of prime divisors. Now choose a prime p not dividing N and fix an isomorphism $B \otimes_{\mathbf{Q}} \mathbf{Q}_p \tilde{\to} M_2(\mathbf{Q}_p)$. Finally, pick a maximal $\mathbf{Z}[1/p]$ order $A \subset B$. The strong approximation theorem tells us that all such A are conjugate in B. The units A^* of A form a discrete subgroup Γ of G. More generally, one can choose a non-maximal $\mathbf{Z}[1/p]$-order A' in A and let Γ' be the units of A'. The groups Γ' form a family of congruence subgroups of Γ.

The main results of p-adic uniformization in this setting say that:

(1) The groups Γ' act discontinuously on the tree X. For A' small enough, this action is free, Γ' is a finitely generated free group, and X/Γ' is a finite graph.
(2) The quotient $S_N(\Gamma') = \mathcal{X}/\Gamma'$ exists as a rigid space; it can be embedded in projective space as a closed rigid subvariety, and therefore is an algebraic curve.

(3) The quotient algebraic curve $S_N(\Gamma')$ is a Shimura curve. It classifies two-dimensional, principally polarized abelian varieties with endomorphism ring equal to a maximal order in the indefinite quaternion algebra with discriminant Np and with level structure determined by $A' \subset A$.
(4) The curve $S_N(\Gamma')$ is totally split over $\mathbf{Q}_p$, meaning that it has a regular model over $\mathbf{Z}_p$ with the property that all of the components of this model are reduced rational curves and all intersection points of components are ordinary double points. The intersection graph of this configuration is exactly X/Γ'.
(5) The genus of $S_N(\Gamma')$ is the genus of the graph X/Γ'. (The genus of a graph is the number of independent cycles in the graph, or more formally the rank of the first homology group of its geometric realization.)

Of particular interest to us are the spaces $\mathcal{O}_{\mathcal{X}}(k)^{\Gamma'}$ where Γ' is a congruence group associated to a quaternion algebra. For $k \geq 2$, elements of this space are "modular forms for Γ'" – that is, functions satisfying the condition

$$f(\gamma z) = (az+c)^k \det(\gamma)^{-k/2} f(z) \qquad \text{for } \gamma \in \Gamma.$$

Such a modular form of (even) weight k corresponds to a global section of the $k/2$-fold tensor power of the canonical bundle $(\Omega^1)^{k/2}$ on $S_N(\Gamma')$.

Proposition 2.4.1. *The residue map* Res : $\mathcal{O}_{\mathcal{X}}(k)^{\Gamma'} \to C_{har}(k)^{\Gamma'}$ *is an isomorphism.*

Proof: The essential point is that the quotient graph X/Γ' has finitely many edges. If we choose finitely many representative edges $e_1, \ldots, e_m$ for this quotient, then the value $c(e)$ of a harmonic function on a general edge is determined by its value on one of these finitely many edges. It follows that the norm $\omega(c)$ is bounded below. In other words, any Γ'-invariant harmonic function is bounded. As a result, we can use Corollary 2.3.4 to construct a preimage for c. This proves surjectivity. When $k = 2$, the space of harmonic functions $C_{har}(2)$ is just the space of harmonic functions on the graph X/Γ', and this is g-dimensional where g is the genus of X/Γ'. On the other hand, the elements of $\mathcal{O}_{\mathcal{X}}(k)^{\Gamma'}$ give rise to holomorphic differential forms on X/Γ', and that space is also g dimensional — therefore the map is injective. When $k > 2$, the space of invariant harmonic cocycles is determined by specifying, on each edge of X/Γ', an element of P_{k-2}; while, for each vertex, one obtains $k-1$ linear relations. Thus the dimension of the space $C_{har}(k)^{\Gamma'}$ is at least $(k-1)(E-V) = (k-1)(g-1)$ where E and V are the number of vertices and edges, respectively in the quotient graph X/Γ'. The Riemann-Roch theorem implies that the space $\mathcal{O}_{\mathcal{X}}(k)^{\Gamma'}$, corresponding to the $k/2$-tensor power of the canonical bundle, has dimension $(k-1)(g-1)$. By dimension counting we see the map is surjective in each case.

2.5. Hecke Operators. The quaternion algebra B has an associated Hecke algebra. Without attempting to work out the whole theory of this algebra, we will indicate the key idea. As above, we let A denote a fixed, maximal $\mathbf{Z}[1/p]$-order, and A' be a sub-$\mathbf{Z}[1/p]$-order of A.

For any unramified prime ℓ of B, the order $A_\ell = A \otimes \mathbf{Z}_\ell$ can be assumed isomorphic to the ring $M_2(\mathbf{Z}_\ell)$. For any ℓ outside of a finite set S containing the ramified primes, we have $A'_\ell = A_\ell$. From the strong approximation theorem and the adelic theory of quaternion algebras (see [**46**, III.4-5]) we see that, for $\ell \notin S$, there are exactly $\ell + 1$ inequivalent left ideals of A' of index ℓ, and these ideals are principal. Let $x_1, \ldots, x_{\ell+1}$ be generators for these ideals. A unit γ in A' permutes these ideals:

$$A' x_i \gamma = A' x_j.$$

Suppose now that $f \in \mathcal{O}_{\mathcal{X}}(k)^{\Gamma'}$. Then

$$(T(\ell)f)(z) = \sum_{i=1}^{\ell+1} x_i(f)$$

is again Γ'-invariant, because, for $\gamma \in \Gamma'$, the left multiplication by g permutes the x_i.

For ℓ outside the finite set S, the operators $T(\ell)$ generate a commutative algebra $\mathbf{T}$ called the Hecke algebra. Since the units $B^\times$ of B act on the tree X through the embedding $B^\times \hookrightarrow \mathrm{GL}_2(\mathbf{Q}_p)$, one obtains an action of $\mathbf{T}$ on the edges of X and therefore on $C_{har}(k)$. Tracing through the definitions, and using the G-equivariance of the residue map, we obtain:

Proposition 2.5.1. *The residue map*

$$\mathcal{O}_{\mathcal{X}}(k)^{\Gamma'} \to C_{har}(k)^{\Gamma'}$$

is a Hecke module isomorphism.

In practical terms, this means that one can compute the Hecke module structure of the spaces of modular forms on the upper half plane by working combinatorially on the tree.

It is also worth noting that the action of the Hecke operators on the finite dimensional spaces $C_{har}(k)^{\Gamma'}$ arises in the theory of classical automorphic forms for B. The matrices representing this action are called *Brandt matrices* and there is extensive literature on them. See [**46**, Exercise II.5.8], as well as the papers by Pizer and collaborators ([**29**]).

3. $\mathcal{L}$-invariants and modular symbols

Now we change course dramatically, and begin a discussion leading to the connection between the global arithmetic of modular forms and the p-adic analysis we've discussed so far in these lectures.

We will rely implicitly on a fairly significant chunk of the theory of classical modular forms. Beyond the basic definitions of modular forms and the theory of Hecke operators, we will make extensive use of the theory of modular symbols and the connection between periods of modular forms and special values of L-functions. The literature on all of these topics is vast. For the foundations, one may consult Shimura's famous book [**44**]. The beginning of the paper [**25**] develops some of the elementary theory of modular symbols and L-functions.

The work of Mazur–Swinnerton-Dyer ([**24**], see also [**25**]) explains how to attach to an eigenform f of even weight k and level M a p-adic L-function $L_p(f, \chi, s)$ that interpolates the "algebraic parts" $L^{alg}(f, \chi, j)$, for $j = 0, \ldots, k-2$ of the special values of the classical L-function of f and its twists by Dirichlet characters χ. The resulting L-function plays a central role in the Iwasawa theory of modular forms and in the p-adic Birch–Swinnerton-Dyer conjecture.

Let χ be a Dirichlet character with $\chi(p) = w = \pm 1$, and let f be an eigenvector for the Hecke operators. In the special case $M = Np$, where N is an integer not divisible by the prime p, and the form f is an eigenvector for the Atkin-Lehner U_p-operator with eigenvalue $a_p = w \cdot p^{(k-2)/2}$, [**25**] showed that the order of vanishing of the p-adic L-function $L_p(f, \omega^{\frac{k-2}{2}}\chi, s)$ at $s = (k-2)/2$ is one higher than that of the classical L-function $L(f, \chi, s)$ at $s = k/2$ (the two functions having different traditional normalizations on the variable s). Here ω is the Teichmuller character. The "exceptional zero conjecture" proposed in [**25**] asserted that there is an invariant $\mathcal{L}(f)$, depending only on the local Galois representation associated to f, such that

$$L_p'(f, \omega^{\frac{k-2}{2}}\chi, (k-2)/2) = \mathcal{L}(f)L(f, \chi, (k-2)/2)^{alg}.$$

When $k = 2$ and f is the modular form associated to an elliptic curve E, the assumption that p precisely divides the level M and that $a_p = \pm 1$ means that E has multiplicative reduction at p. In that case, [**25**] presented numerical evidence that

$$\mathcal{L}(f) = \frac{\log(q)}{\operatorname{ord}(q)}$$

where q is the Tate period of the elliptic curve E at p and log is the p-adic logarithm. This weight 2 form of the conjecture was proved by Greenberg–Stevens ([**19**]) using Hida theory.

In the higher weight case, in the period since [**25**], a number of different candidates for the invariant $\mathcal{L}(f)$ have been proposed. (See [**7**] for more background and references.) These include:

(1) An invariant $\mathcal{L}_T(f)$ built by taking advantage of the theory of p-adic uniformization of Shimura curves — we will discuss this in more detail later;
(2) An invariant $\mathcal{L}_C(f)$ built using Coleman's theory of p-adic integration on modular curves;
(3) An invariant $\mathcal{L}_{FM}(f)$ due to Fontaine–Mazur built using Fontaine's classification of p-adic representations;
(4) An invariant $\mathcal{L}_O(f)$ due to Darmon (in weight two) and Orton (in general) using "modular form-valued distributions" (also to be discussed later in this lecture);
(5) An invariant $\mathcal{L}_B(f)$ due to Breuil that derives from his investigations of p-adic Langlands theory (discussed in the next lecture).

All of these invariants are known to be equal:

(1) $\mathcal{L}_T = \mathcal{L}_{FM} = \mathcal{L}_C$ by Coleman-Iovita ([**5**]) and Iovita-Spiess ([**21**]).
(2) $\mathcal{L}_O = \mathcal{L}_B$ by Breuil ([**2**])
(3) $\mathcal{L}_B = \mathcal{L}_{FM}$ by Colmez. ([**8**]).
(4) $\mathcal{L}_O = \mathcal{L}_T$ by Bertolini, Darmon, and Iovita ([**1**]).

The Exceptional Zero Conjecture itself has been proved in general by Stevens (for the Coleman invariant), by Kato, Kurihara and Tsuji (for the Fontaine–Mazur invariant), by Darmon and Orton for $\mathcal{L}_O(f)$, by Emerton using Breuil's invariant, and by Bertolini–Darmon–Iovita using $\mathcal{L}_T(f)$. Stevens's and the Kato–Kurihara–Tsuji result remain unpublished, but one can consult [**9**] for information. For the other results, see [**14**], [**28**],[**11**], and [**1**].

3.1. $\mathcal{L}_T(f)$ and p-adic uniformization. As an application of the theory developed in Lectures I and II, let us describe the construction of the invariant $\mathcal{L}_T(F)$ when F is a modular form for a Shimura curve. In other words, we are in the situation of section 2.4. We begin with a definite quaternion algebra B of discriminant N and a prime p not dividing N. Let A' be an order contained in a fixed maximal $\mathbf{Z}[1/p]$-order A in B, and let Γ' be the discrete group of units in A'. Let F be a modular form of weight k (k even and $k \geq 2$) for Γ'. Assume that F is an eigenform for the Hecke algebra of the quaternion algebra B.

From our integration theory, associated to this F we have a distribution λ_F on $C^{an}(K, 2-k)/P_{k-2}$. We define two elements of $H^1(\Gamma', \mathrm{Hom}(P_{k-2}, \mathbf{C}_p))$ using this distribution. Fix any point $z \in \mathcal{X}(\mathbf{C}_p)$ and set :

$$h^F_{\log}(\gamma, P(x)) = \lambda_F\left(P(x)\log\left(\frac{x-\gamma(z)}{x-z}\right)\right)$$

$$h^F_{\mathrm{ord}}(\gamma, P(x)) = \lambda_F\left(P(x)\,\mathrm{ord}\left(\frac{x-\gamma(z)}{x-z}\right)\right).$$

The functions $P(x)\log(\frac{x-\gamma(z)}{x-z})$ and $P(x)\,\mathrm{ord}(\frac{x-\gamma(z)}{x-z})$ both belong to $C^{an}(K, 2-k)$, since both are locally analytic and have the correct pole order at infinity. The fact that $h^F_{\log}$ and $h^F_{\log}$ are cocycles that depend on z only up to a coboundary is a straighforward calculation.

One can interpret $h^F_{\log}$ as a period of the form $F(z)dz$ on $S_N(M)$. Using the expression of F as a Poisson integral, we have (formally):

$$h^F_{\log}(\gamma, P(x)) = \int_z^{\gamma(z)} \int_{\mathbf{P}^1} \frac{1}{z-x} d\lambda_F$$

as follows from a change in the order of integration. Using the theory of Coleman integration, one can give meaning to this integral, and in fact this argument is legitimate — see [**45**].

Similarly, one can interpret h^F_{ord} as a period on the tree. Choose z so that $r(z)$ is a vertex v on the tree. Then one has the following.

Lemma 3.1.1. *The integral defining h^F_{ord} reduces to a sum on the tree:*

$$h^F_{\mathrm{ord}}(\gamma, P(x)) = \sum_{v \mapsto \gamma(v)} c_e(P(x))$$

where the sum is over the edges e on the minimal path joining v to $\gamma(v)$.

Proof. Let e be an oriented edge in the tree X, and let $s, t \in \mathcal{X}$ be points whose reductions are the source and terminal vertices of e, respectively. Then for $u \in \mathbf{P}^1(K)$, one readily verifies that

$$\mathrm{ord}\left(\frac{u-t}{u-s}\right) = \begin{cases} -1 & \text{if } u \in U(\overline{e}) \\ 0 & \text{otherwise,} \end{cases}$$

where $U(\overline{e})$ is the open subset of $\mathbf{P}^1(K)$ associated to the oppositely oriented edge of e (see Section 1.3.4). Choosing points $z = z_0, z_1, \ldots, z_n = \gamma(z)$ reducing to successive vertices on the path from z to $\gamma(z)$ we obtain

$$\begin{aligned} h_{\mathrm{ord}}^F(\gamma, P(x)) &= \sum_{i=1}^{n} \lambda_F \left(P(x) \operatorname{ord} \left(\frac{x - z_{i+1}}{x - z_i} \right) \right) \\ &= -\sum_{i=1}^{n} c(\overline{e_i})(P(x)) \\ &= \sum_{i=1}^{n} c(e_i)(P(x)) \end{aligned}$$

where e_i joins the reductions of z_i and z_{i+1}. □

THEOREM 3.1.2. *(Schneider, DeShalit) The two maps* $h_{\mathrm{ord}} : F \mapsto h_{\mathrm{ord}}^F$ *and* $h_{\log} : F \mapsto h_{\log}^F$ *are homomorphisms*

$$\mathcal{O}(k)^{\Gamma'} \to H^1(\Gamma', \mathrm{Hom}(P_{k-2}, \mathbf{C}_p))$$

commuting with the natural action of the Hecke algebra $\mathbf{T} \otimes \mathbf{C}_p$ *on both sides. Furthermore,* h_{ord} *is an isomorphism.*

PROOF. See [**10**] and [**31**]. □

With this theorem, we can construct the invariant $\mathcal{L}_T(F)$. Theorem 3.1.2 and the fact that the F-isotypic component of $\mathcal{O}(k)^{\Gamma'}$ is 1-dimensional yields:

DEFINITION 3.1.3. There is a unique $\mathcal{L}_T(F) \in \mathbf{C}_p$ such that

$$h_{\log}^F - \mathcal{L}_T(F) h_{\mathrm{ord}}^F = 0$$

in $H^1(\Gamma', \mathrm{Hom}(P_{k-2}, \mathbf{C}_p))$. This is called the $\mathcal{L}_T$–invariant of the form F.

3.2. Modular Symbols. To develop the additional theory of $\mathcal{L}$-invariants following Breuil and Darmon, we must undertake a digression into the theory of modular symbols, and also develop some of the ideas of Darmon's integration on $\mathcal{X} \times \mathcal{H}$, where $\mathcal{H}$ is the classical upper half plane. We follow, in part, Breuil's presentation ([**2**]) in this discussion.

Fix a normalized newform f of even weight $k \geq 2$ on $\Gamma_0(M)$ for some integer M. We assume that $T_\ell f = a_\ell f$ for $(\ell, M) = 1$. The eigenvalues a_ℓ generate an extension E of $\mathbf{Q}$ with ring of integers R. We will view E as a subfield of $\mathbf{C}_p$.

In working with these formulae, one caveat is necessary. It's traditional in the theory of modular forms to work with the right action (the "slash" action) on modular forms given by the formula:

$$f(z)|_g = (cz+d)^{-k} \det(g)^{k/2-1} f\left(\frac{az+b}{cz+d}\right).$$

Since we have consistently worked with left actions, we use the associated left action

$$g(f)(z) = f(z)|_{g^t}$$

where g^t is the transpose of g.

The following theorem of Shimura is the starting point of the theory we will describe.

THEOREM 3.2.1. *There are nonzero periods $\Omega_f^{\pm} \in \mathbf{C}$ such that, for any $P(z) \in P_{k-2}(R)$ and any rational number r, we have*

$$\left(\int_r^\infty f(z)P(z)dz\right)^{\pm} := \pi i \left(\int_r^\infty f(z)P(z)dz \pm \int_{-r}^\infty f(z)P(-z)dz\right) \in R\Omega_f^{\pm} \in \mathbf{C}.$$

Let D be the set of divisors on $\mathbf{P}^1(\mathbf{Q})$, and let D_0 be the subspace of divisors of degree zero. We associate to our form f the "modular symbol"

$$\phi_f^{\pm} \in \mathrm{Hom}(D_0, \mathrm{Hom}(P_{k-2}(E), E))$$

by defining

$$\begin{aligned}
\phi_f^{\pm}([r]-[s])(P) &:= \frac{1}{\Omega_f^{\pm}}\left(\int_s^r f(z)P(z)dz\right)^{\pm} \\
&= \frac{1}{\Omega_f^{\pm}}\left(\int_s^\infty f(z)P(z)dz\right)^{\pm} - \frac{1}{\Omega_f^{\pm}}\left(\int_r^\infty f(z)P(z)dz\right)^{\pm}.
\end{aligned}$$

The modular symbol enjoys the following invariance property for $g \in \Gamma_0(M)$:

$$\begin{aligned}
\phi_f^{\pm}([g(r)]-[g(s)])(P) &= \left(\int_{g(r)}^{g(s)} f(z)P(z)dz\right)^{\pm} \\
&= \left(\int_r^s f(g^{-1}(z))P(g^{-1}(z))dg^{-1}(z)\right)^{\pm} \\
&= \left(\int_r^s (-bz+a)^{k-2} f(z)P(\frac{dz-c}{-bz+a})dz\right)^{\pm} \\
&= \phi_f^{\pm}([r]-[s])(g^{-1}(P)(z))^{\pm}
\end{aligned}$$

so that

$$\phi_f^{\pm} \in \mathrm{Hom}(D_0, \mathrm{Hom}(P_{k-2}, E))^{\Gamma_0(M)}.$$

Theorem 3.2.1 implies that for any element $[r]-[s]$ of D_0, the corresponding linear form

$$\phi_f([r]-[s]) \in \mathrm{Hom}(P_{k-2}(E), E)$$

is bounded.

3.2.1. *Modular symbols and L-values.* Both the algebraic part of the L-function associated to a modular form and its p-adic L-function may be expressed in terms of modular symbols.

DEFINITION 3.2.2. The algebraic part of the special value(s) of the classical L-function associated to f and a Dirichlet character χ of conductor c is given by the formula

$$L^{alg}(f, \chi, j) = \frac{c^{j+1} j!}{(-2\pi i)^j \tau(\overline{\chi}) \Omega_f^{w_\infty}} L(f, \overline{\chi}, j+1),$$

where $w_\infty = \chi(-1)$.

A computation using the expression for the L-function of f as the Mellin transform of the modular form f yields the following formula expressing L^{alg} in terms of modular symbols.

LEMMA 3.2.3. *We have*

$$L^{alg}(f,\chi,j) = \sum_{\nu\in(\mathbf{Z}/c\mathbf{Z})^\times} \chi(\nu)\phi_f^{w_\infty}\left(\left[\frac{-\nu}{c}\right]-[\infty]\right)\left((cz+\nu)^j\right)$$

for $j=0,\dots,(k-2)/2$.

PROOF. See [**25**, Section 8]. $\square$

Next, we briefly recall the construction of the p-adic L-function from [**25**]. Let

$$\mathbf{Z}_{p,c} := \varprojlim \mathbf{Z}/p^n c\mathbf{Z} \cong \mathbf{Z}_p \times \mathbf{Z}/c\mathbf{Z}.$$

For $x\in\mathbf{Z}_{p,c}$, let x_p denote the projection of x to $\mathbf{Z}_p$. For $a\in\mathbf{Z}_{p,c}^\times$, write $D(a,r) := a+cp^r \subset \mathbf{Z}_{p,c}^\times$.

The p-adic L-function is constructed from the (unique) distributions $\mu^\pm_{f,\mathrm{MTT}}$ on the space of locally analytic functions on $\mathbf{Z}_{p,c}^\times$ satisfying

$$\int_{D(a,r)} P(x_p)d\mu^\pm_{f,\mathrm{MTT}}(x) = \left(wp^{\frac{k-2}{2}}\right)^{-r}\phi_f^\pm\left([\infty]-\left[\frac{a}{p^r c}\right]\right)(P(p^rcz+a)) \tag{3.2.4}$$

for all polynomials of P degree at most $(k-2)/2$. Here and in the sequel, the left side is shorthand notation for

$$\mu^\pm_{f,MTT}\left(\delta_{D(a,r)}(x)P(x_p)\right),$$

where $\delta_{D(a,r)}$ denotes the characteristic function of the open set $D(a,r)$.

The boundedness properties of the modular symbols imply that there is a unique distribution on locally analytic functions on $\mathbf{Z}_{p,c}$ that, restricted to locally polynomial functions, satisfies the condition in equation 3.2.4. This is another instance of the p-adic integration theory that we referred to in Theorem 2.3.2. For details of the construction, see [**25**, Section 11].

Let $\chi:\mathbf{Z}_{p,c}^\times\to\mathbf{C}_p^\times$, and define $\langle\cdot\rangle:\mathbf{Z}_{p,c}^\times\to 1+p\mathbf{Z}_p$ by $\langle x\rangle := x_p/\omega_{\mathrm{Teich}}(x)$, where $\omega_{\mathrm{Teich}}(x)$ is the p-adic Teichmuller character. The p-adic L-functions attached to f and χ are defined as follows:

$$L_p^\pm(f,\chi,s) := \int_{\mathbf{Z}_{p,c}^\times}\chi(x)\langle x\rangle^s d\mu^\pm_{f,\mathrm{MTT}}(x).$$

If $\epsilon=\chi(-1)\cdot(-1)^{\frac{k-2}{2}}=\pm1$, then $L_p^{-\epsilon}(f,\chi,s)=0$ and $L_p^\epsilon(f,\chi,s)$ will be *a prioiri* non-trivial. Thus writing $L_p=L_p^+ + L_p^-$, we see that $L_p=L_p^\epsilon$.

3.2.2. *Modular symbols and the tree.* Darmon introduced the remarkable idea of blending p-adic integration and the p-adic upper half plane with classical modular forms in his paper [**11**]. Assume that the level M of the preceding section can be written $M=Np$ with $(N,p)=1$. Define

$$\Gamma_0^p(N) = \left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in \mathrm{SL}_2(\mathbf{Z}[1/p]),\ c\equiv 0 \pmod N\right\}.$$

Similarly, let

$$\tilde\Gamma_0^p(N) = \left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in \mathrm{GL}_2(\mathbf{Z}[1/p])^+,\ c\equiv 0 \pmod N\right\}$$

where $\mathrm{GL}_2(\mathbf{Q})^+$ is the group of invertible matrices with positive determinant.

DEFINITION 3.2.5. Let $\mathcal{H}$ be the classical upper half plane over $\mathbf{C}$. Define $S_k^0(X, \Gamma_0^p(N))$ to be the complex vector space of "harmonic, modular form-valued" functions

$$F : \mathrm{Edges}(\mathcal{X}) \times \mathcal{H} \to \mathbf{C}$$

satisfying the following conditions:

(1) $F(\gamma e, z) = \gamma(F(e,z)) = F(e,z)|_{\gamma^t}$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^p(N)$.
(2) $F(e', z) = -F(e,z)$ where e' is the edge opposite to e.
(3) $\sum_{e \mapsto v} F(e,z) = 0$ where the sum is over the edges leaving v.
(4) Each $F(e, \cdot)$ is a cusp form of weight k on $\mathcal{H}$ for the group

$$\Gamma_e = \{\gamma \in \Gamma_0^p(N) : \gamma e = e\}.$$

The group $\mathrm{GL}_2(\mathbf{Q})^+$ acts on the left on $S_k^0(\mathcal{X}, \Gamma_0^p(N))$ via the formula

$$g(F)(e,z) = g(F(g^{-1}e, z)) = (bz+d)^{-k} \det(g)^{k/2-1} F\left(g^{-1}e, \frac{az+c}{bz+d}\right).$$

The space $S_k^0(X, \Gamma_0^p(N))$ is quite small. Notice that, if $F \in S_k^0(X, \Gamma_0^p(N))$, then the restriction $F(e_0, \cdot)$ of F to the basic edge e_0 satisfies

$$\gamma(F(e_0, z)) = F(e_0, z)$$

for all

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^p(N) \cap \Gamma_{e_0} = \Gamma_0(Np).$$

In other words, $F(e_0, \cdot)$ is a cusp form for $\Gamma_0(M) = \Gamma_0(Np)$.

PROPOSITION 3.2.6. *The restriction map*

$$S_k^0(X, \Gamma_0^p(N)) \to S_k(\Gamma_0(Np), \mathbf{C})$$

is injective and has image equal to the subspace of forms that are "new at p."

Proof: See [**28**, Section 2.1]. The point is that the harmonicity requirement amounts to the statement that the form has to be in the kernel of the trace map(s) from forms of level Np to forms of level N. $\square$

Given a p-new form f, we can find a corresponding element of $S_k^0(X, \Gamma_0^p(N))$ by defining

$$F(ge_0, z) = g(f(z)) = w^{\mathrm{ord}(\det(ad-bc))}(bz+d)^{-k} \det(g)^{k/2-1} f\left(\frac{az+c}{bz+d}\right)$$

for $g \in \tilde{\Gamma}$, where w is the sign such that $W_p(f) = -wf$ for the Atkin-Lehner operator W_p.

3.2.3. *Modular symbols, harmonic cocycles, and distributions.* Let f be a cusp form of level M that is new at p, and let F be the element of $S_k^0(X, \Gamma_0^p(N))$ associated to f by Proposition 3.2.6. Define a harmonic function Φ_f with values in $\mathrm{Hom}(P_{k-2}(E), E)$ on the edges of X by the rule

$$\Phi_f^{\pm}([r]-[s])(e)(P) := \phi_{F(e,\cdot)}^{\pm}([r]-[s])(P).$$

PROPOSITION 3.2.7. *(Orton) The harmonic function $\Phi_f^{\pm}([r]-[s])$ is bounded.*

Proof: We need to verify that

$$\omega(\Phi^{\pm}([r]-[s])(\gamma e)(\gamma P)) \geq N$$

for some fixed integer N and polynomials P with coefficients in R. But

$$\begin{aligned} \phi^{\pm}_{F(\gamma e,\cdot)}([r]-[s])(P) &= \pm\left(\int_r^s \gamma(f)(z)\gamma(P)(z)dz\right)^{\pm} \\ &= \pm\left(\int_{\gamma(r)}^{\gamma(s)} f(z)P(z)dz\right)^{\pm} \end{aligned}$$

and this is bounded by Theorem 3.2.1. □

From this boundedness result, we obtain from $\Phi^{\pm}_f([r]-[s])$ a distribution $\lambda^{\pm}_f([r]-[s])$ on $C^{la}(K,2-k)$ that extends to $C^{an}(K,2-k)$ following the procedure discussed in Section 2.3.

Let $\mathcal{M} := \mathrm{Hom}(D_0, \mathrm{Hom}(P_{k-2}(\mathbf{C}_p), \mathbf{C}_p))$, the space of modular symbols valued in the dual of the space of polynomials of degree at most $k-2$. Then choosing any $a \in \mathcal{X}$, we obtain maps

$$\{\text{Cusp forms of level } Np \text{ new at } p\} \to H^1(\Gamma_0^p(N), \mathcal{M})$$

defined by

$$lc^{\pm}_f(\gamma)([r]-[s])(P) = \lambda^{\pm}_f([r]-[s])\left(P(x)\log\left(\frac{x-\gamma a}{x-a}\right)\right) \tag{3.2.8}$$

and

$$oc^{\pm}_f(\gamma)([r]-[s])(P) = \lambda^{\pm}_f([r]-[s])\left(P(x)\,\mathrm{ord}\left(\frac{x-\gamma a}{x-a}\right)\right). \tag{3.2.9}$$

The cohomology classes of these maps are independent of the choice of a.

3.3. Orton's L-invariant. The difficulty with the invariant $\mathcal{L}_T$ is that it is only indirectly related to the Mazur-Swinnerton-Dyer p-adic L-function that plays a role in the exceptional zero conjecture. This is because the p-adic L-function is constructed using a modular form on the usual upper half plane corresponding to a usual modular curve, while the construction of the $\mathcal{L}_T$ invariant uses a Shimura curve. The connection between these two constructions comes from the Jacquet-Langlands lifting theorem, which asserts that there is a correspondence between modular forms on Shimura curves and certain modular forms on classical modular curves. In fact, not all forms on modular curves come from Shimura curves, and so the invariant $\mathcal{L}_T$ isn't even defined for a modular form on $\Gamma_0(N')$, with N' general.

The invariant $\mathcal{L}_O$ constructed by Darmon and Orton is a hybrid object that mixes p-adic uniformization with classical modular forms. It's construction has something of the same flavor as that of $\mathcal{L}_T$, but it is directly connected to both the p-adic L-function and the classical L-function of a form f on $\Gamma_0(N)$. In this section we will construct Orton's invariant (see Definition 3.3.4) and relate it to L-values.

3.3.1. *Cohomology of Modular Symbols.* For each prime $\ell \nmid N$, we define an action of the Hecke operator T_ℓ on $H^1(\Gamma_0^p(N), \mathcal{M})$. Let $\{\delta_j\}_{j=0}^{\ell}$ be a set of matrices in $\mathrm{GL}_2(\mathbf{Q})$ such that

$$\Gamma_0^p(N)\begin{pmatrix}1 & 0\\ 0 & \ell\end{pmatrix}\Gamma_0^p(N) = \bigsqcup_{j=0}^{\ell} \Gamma_0^p(N)\delta_j.$$

For each $\gamma \in \Gamma_0^p(N)$ and $j = 0, \ldots, \ell$, there exists a unique $\gamma_j \in \Gamma_0^p(N)$ and index $i(\gamma, j)$ such that $\delta_j \gamma = \gamma_j \delta_{i(\gamma,j)}$. Let $\tilde{c}$ be a cohomology class in $H^1(\Gamma_0^p(N), \mathcal{M})$ represented by a cocycle c. The cohomology class represented by the cocycle

$$T_\ell(c)(\gamma) := \ell^{\frac{k-2}{2}} \sum_{j=0}^{\ell} \delta_j^{-1} c(\gamma_j)$$

is independent of choices, and is defined to be $T_\ell(\tilde{c})$.

We now define an Atkin-Lehner involution "at infinity." Let $\alpha_\infty = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. The operator W_∞ on $H^1(\Gamma_0^p(N), \mathcal{M})$ is defined by

$$W_\infty(c)(\gamma) := \alpha_\infty c(\alpha_\infty \gamma \alpha_\infty).$$

Definition 3.3.1. Let V be a space endowed with an action of the Hecke algebra $\mathbf{T}$. For a sign $w_\infty = \pm 1$, let V^{f, w_∞} denote the space of elements $v \in V$ such that $T_\ell(v) = a_\ell \cdot v$ for each $\ell \nmid N$ and $W_\infty(v) = w_\infty \cdot v$, where a_ℓ denotes the eigenvalue of f for the Hecke operator T_ℓ.

In the next section, we will prove:

Lemma 3.3.2. *For each $w_\infty = \pm 1$, the cohomology group $H^1(\Gamma_0^p(N), \mathcal{M})^{f, w_\infty}$ is a 1-dimensional $\mathbf{C}_p$-vector space.*

Lemma 3.3.3. *For each $w_\infty = \pm 1$, we have*

$$lc_f^{w_\infty}, oc_f^{w_\infty} \in H^1(\Gamma_0^p(N), \mathcal{M})^{f, w_\infty}.$$

Proof. (Sketch; see [**28**, Lemma 5.3]) The fact that $lc_f^{w_\infty}, oc_f^{w_\infty}$ are in the f, w_∞-isotypic subspace of $H^1(\Gamma_0^p(N), \mathcal{M})$ follows from the corresponding fact for Φ. More precisely, one can show that

$$\ell^{(k-2)/2} \sum_{j=0}^{\ell} \Phi_f^{w_\infty}([\delta_j x] - [\delta_j y])(\delta_j e)(P|_{\delta_j^{-1}}) = a_\ell \Phi_f^{w_\infty}([x] - [y])(e)(P)$$

and

$$\Phi_f^{w_\infty}([\alpha_\infty x] - [\alpha_\infty y])(\alpha_\infty e)(P|_{\alpha_\infty^{-1}}) = w_\infty \Phi_f^{w_\infty}([x] - [y])(e)(P)$$

from the corresponding formulas for $\phi_f^{w_\infty}$. $\square$

In Corollary 3.3.22 we will show that $oc_f^{w_\infty} \neq 0$. In view of Lemmas 3.3.2 and 3.3.3, we therefore propose:

Definition 3.3.4. For each $w_\infty = \pm 1$, define $\mathcal{L}_O^{w_\infty} \in \mathbf{C}_p$ by the equality

$$lc_f^{w_\infty} = \mathcal{L}_O^{w_\infty} \cdot oc_f^{w_\infty}.$$

The goal of the remainder of this section is to prove Lemma 3.3.2. To simplify the notation, let $V := \mathrm{Hom}(P_{k-2}(\mathbf{C}_p), \mathbf{C}_p)$. Applying $\mathrm{Hom}(-, V)$ to the short exact sequence

$$0 \to D_0 \to D \to \mathbf{Z}$$

defining D_0, we obtain

$$0 \to V \to \mathcal{F} \to \mathcal{M} \to 0, \tag{3.3.5}$$

where $\mathcal{F} := \mathrm{Hom}(D, V)$ and $\mathcal{M} := \mathrm{Hom}(D_0, V)$.

Consider the long exact sequence arising from (3.3.5) by taking cohomology for $\Gamma_0(N)$. The first term $V^{\Gamma_0(N)}$ is trivial when $k > 2$ and equal to $\mathbf{C}_p$ when $k = 2$ [**20**, p. 165 Lemma 2]. Furthermore, $H^2(\Gamma_0(N), V) = 0$ [**20**, p. 162, Prop. 1].

For each cusp of $\Gamma_0(N)$, i.e. for each class in $\Gamma_0(N)\backslash\mathbf{P}^1(\mathbf{Q})$, we choose a representative $x \in \mathbf{P}^1(\mathbf{Q})$ and let $\Gamma_0(N)_x$ denote the stabilizer of x in $\Gamma_0(N)$. The module $\mathcal{F}$ is easily seen to be a sum of induced modules:

$$\mathcal{F} = \bigoplus_x \operatorname{Ind}_{\Gamma_0(N)_x}^{\Gamma_0(N)} V.$$

By Shapiro's Lemma, we therefore have

$$H^i(\Gamma_0(N), \mathcal{F}) = \bigoplus_x H^i(\Gamma_0(N)_x, V).$$

In the long exact sequence associated to (3.3.5), the map

$$H^1(\Gamma_0(N), V) \to H^1(\Gamma_0(N), \mathcal{F}) \cong \bigoplus_x H^1(\Gamma_0(N)_x, V)$$

is simply the direct sum of restriction maps; its kernel is called the parabolic cohomology group, and denoted $H^1_{\mathrm{par}}(\Gamma_0(N), V)$. We thus obtain two exact sequences:

$$0 \to \left(\bigoplus_x V^{\Gamma_0(N)_x}\right)/V^{\Gamma_0(N)} \to \mathcal{M}^{\Gamma_0(N)} \to H^1_{\mathrm{par}}(\Gamma_0(N), V) \to 0 \tag{3.3.6}$$

and

$$\begin{aligned} 0 \to H^1_{\mathrm{par}}(\Gamma_0(N), V) \to H^1(\Gamma_0(N), V) \to \\ \bigoplus_x H^1(\Gamma_0(N)_x, V) \to H^1(\Gamma_0(N), \mathcal{M}) \to 0. \end{aligned} \tag{3.3.7}$$

The key results we will use to study these sequences are the classical Eichler-Shimura isomorphisms, which state [**20**, Section 6.2]:

$$H^1_{\mathrm{par}}(\Gamma_0(N), V) \cong S_k(\Gamma_0(N)) \oplus \overline{S_k(\Gamma_0(N))} \tag{3.3.8}$$

$$H^1(\Gamma_0(N), V) \cong S_k(\Gamma_0(N)) \oplus \overline{S_k(\Gamma_0(N))} \oplus E_k(\Gamma_0(N)). \tag{3.3.9}$$

The sequences (3.3.6) and (3.3.7) are Hecke equivariant, as are the Eichler-Shimura isomorphisms. The Hecke structure of the module on the left in (3.3.6) is given by the action of the Hecke operators on the cusps of $\Gamma_0(N)$; therefore it is not surprising that [**28**, §7.2]:

LEMMA 3.3.10. *We have an isomorphism of Hecke modules:*

$$\left(\bigoplus_x V^{\Gamma_0(N)_x}\right)/V^{\Gamma_0(N)} \cong E_k(\Gamma_0(N)).$$

We leave the proof to the reader, as well as that of:

LEMMA 3.3.11. *Given a Hecke equivariant short exact sequence of finite dimensional $\mathbf{C}_p$-vector spaces $0 \to V_1 \to V_2 \to V_3 \to 0$ with $V_1^f = 0$, the map $V_2 \to V_3$ induces an isomorphism $V_2^f \to V_3^f$. Alternatively, if $V_3^f = 0$, then the map $V_1 \to V_2$ induces an isomorphism $V_1^f \to V_2^f$.*

PROPOSITION 3.3.12. *We have $\mathcal{M}^{\Gamma_0(N),f} = 0$.*

PROOF. This follows from the previous two lemmas, sequence (3.3.6), and the Eichler-Shimura isomorphism (3.3.8), since f is a form of level Np which is not old at p. $\square$

Arguing similarly for N replaced by Np, we find:

PROPOSITION 3.3.13. *For each $w_\infty = \pm 1$, the space $\mathcal{M}^{\Gamma_0(Np),f,w_\infty}$ is a 1-dimensional $\mathbf{C}_p$-vector space.*

Turning now to sequence (3.3.7), we note that each group $\Gamma_0(N)_x$ is infinite cyclic, generated by an element denoted π_x. Thus $H^1(\Gamma_0(N)_x, V) = V/(\pi_x - 1)V$. One checks (see [**20**, p. 166 (2a)], for example) that this is a 1-dimensional $\mathbf{C}_p$-vector space. A dimension count in (3.3.7) using the Eichler-Shimura isomorphisms shows that $H^1(\Gamma_0(N), \mathcal{M})$ is trivial when $k > 2$, and has dimension 1 when $k = 2$; in either case the module is Eisenstein, so we obtain:

PROPOSITION 3.3.14. *We have $H^1(\Gamma_0(N), \mathcal{M})^f = 0$.*

We are now in a position to prove Lemma 3.3.2. The group $\Gamma_0^p(N)$ is the amalgamation of the groups $\Gamma_0(N)$ and its conjugate $\Gamma_0(N)' := \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^{-1} \Gamma_0(N) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$, with respect to their intersection $\Gamma_0(Np)$. This fact follows from the fact that a fundamental domain for the action of $\Gamma_0^p(N)$ on the tree X is given by the single edge e_0 with stabilizer $\Gamma_0(Np)$, and its two boundary vertices with stabilizers $\Gamma_0(N)$ and $\Gamma_0(N)'$. From this amalgamation property, one deduces an exact sequence (see [**43**, §2.6]):

$$0 \longrightarrow \mathcal{M}^{\Gamma_0^p(N)} \longrightarrow \mathcal{M}^{\Gamma_0(N)} \oplus \mathcal{M}^{\Gamma_0(N)'} \longrightarrow \mathcal{M}^{\Gamma_0(Np)} \longrightarrow H^1(\Gamma_0^p(N), \mathcal{M}) \longrightarrow H^1(\Gamma_0(N), \mathcal{M}) \oplus H^1(\Gamma_0(N)', \mathcal{M}).$$

By breaking this exact sequence into short exact sequences, we find from Lemma 3.3.11, Proposition 3.3.12 and Proposition 3.3.14 that:

$$\mathcal{M}^{\Gamma_0^p(N),f} = 0 \quad \text{and} \quad (\mathcal{M}^{\Gamma_0(Np)})^{f,w_\infty} \xrightarrow{\sim} H^1(\Gamma_0^p(N), \mathcal{M})^{f,w_\infty}. \tag{3.3.15}$$

Proposition 3.3.13 then concludes the proof of Lemma 3.3.2.

3.3.2. *Specializations of the cohomology classes.* Fix a positive integer c and an integer ν relatively prime to c. The pair (c, ν) gives rise to a $\mathbf{Q}$-algebra embedding $\Psi : \mathbf{Q} \times \mathbf{Q} \to M_2(\mathbf{Q})$ via the formula

$$\Psi(1,0) = \begin{pmatrix} 1 & \nu/c \\ 0 & 0 \end{pmatrix}.$$

Let s be the order of p^2 in $(\mathbf{Z}/c\mathbf{Z})^\times$. The group $\Psi(\mathbf{Q}^\times \times \mathbf{Q}^\times) \cap \Gamma_0^p(N)$ is an infinite cyclic group, generated by

$$\gamma_\Psi := \begin{pmatrix} p^s & (p^s - p^{-s})\nu/c \\ 0 & p^{-s} \end{pmatrix}.$$

The fixed points of γ_Ψ are $x_\Psi = \infty$ and $y_\Psi = -\nu/c$, and the polynomial

$$P_\Psi(z) = (cz + \nu)^{\frac{k-2}{2}}$$

is fixed by γ_Ψ as well.

DEFINITION 3.3.16. Define

$$LI_\Psi^\pm := lc_f^\pm(\gamma_\Psi)([x_\Psi]-[y_\Psi])(P_\Psi) \tag{3.3.17}$$

and

$$W_\Psi^\pm := oc_f^\pm(\gamma_\Psi)([x_\Psi]-[y_\Psi])(P_\Psi). \tag{3.3.18}$$

Since γ_Ψ fixes x_Ψ, y_Ψ, and P_Ψ, one easily checks that $b(\gamma_\Psi)([x_\Psi]-[y_\Psi])(P_\Psi)=0$ for a coboundary b; thus the equations for LI_Ψ and W_Ψ are well-defined. As we now explain, the values LI_Ψ and W_Ψ encode the central critical values of the p-adic and classical L-functions attached to f. As usual, let χ be a Dirichlet character of conductor c with $\chi(p)=w$ and $\chi(-1)=w_\infty$.

THEOREM 3.3.19. *With notation as above, we have*

$$L^{alg}\left(f,\chi,\frac{k-2}{2}\right)=\frac{1}{2s}\sum_{\nu\in(\mathbf{Z}/c\mathbf{Z})^\times}\chi(\nu)W_\Psi^{w_\infty} \tag{3.3.20}$$

and

$$L_p'\left(f,\omega^{\frac{k-2}{2}}\chi,\frac{k-2}{2}\right)=\frac{1}{2s}\sum_{\nu\in(\mathbf{Z}/c\mathbf{Z})^\times}\chi(\nu)LI_\Psi^{w_\infty}. \tag{3.3.21}$$

Consequently,

$$L_p'(f,\omega^{\frac{k-2}{2}}\chi,(k-2)/2)=\mathcal{L}_O^{w_\infty}(f)L(f,\chi,(k-2)/2)^{alg}.$$

(This is the "exceptional zero conjecture" (for weight $k\geq 2$) as originally posed in [**25**, Section 15]*).*

PROOF. The proofs of equations (3.3.20) and (3.3.21) involve calculations on the tree. This will take the next few sections. $\square$

COROLLARY 3.3.22. $oc_f^{w_\infty}\neq 0$.

PROOF. A result of Rohrlich [**30**] implies that there is a Dirichlet character χ as desired such that $L(f,\chi,k/2)\neq 0$. $\square$

3.3.3. *First part of proof of Orton's Theorem.* The first step in proving Orton's theorem is to evaluate W_Ψ. We begin with an explicit evaluation of the right side of equation (3.2.9), which defines a cocycle representing the cohomology class oc_f.

LEMMA 3.3.23. *Suppose that $a\in\mathcal{X}$ reduces to a vertex v of the tree X. Then we have*

$$oc_f^\pm(\gamma)([r]-[s])(P)=\sum_{e\in(v\to\gamma v)}\Phi_f^\pm([r]-[s])(e)(P),$$

where $(v\to\gamma v)$ represents the unique path in X from the vertex v to the vertex γv, and the sum on the right side is indexed by the oriented edges e in this path.

PROOF. This is the same argument that we used in Lemma 3.1.1. $\square$

For each $\nu\in(\mathbf{Z}/c\mathbf{Z})^\times$, let J_ν denote the coset $\nu\langle p\rangle\subset(\mathbf{Z}/c\mathbf{Z})^\times$. Let s' denote the order of p modulo c, so $s=s'$ if s' is odd and $s=s'/2$ if s' is even. For $a\in J_\nu$, denote by $j(a)$ the equivalence class mod s' such that $a\equiv\nu p^{j(a)} \pmod c$. Note that the expression $w^{j(a)}$ is well-defined if either $w=1$ or s' is even.

PROPOSITION 3.3.24. *We have*

$$W_\Psi^\pm = \beta \sum_{a \in J_\nu} w^{j(a)} \phi_f^\pm \left(\left[-\frac{a}{c} \right] - [\infty] \right) \left((cz+a)^{(k-2)/2} \right),$$

where

$$\beta = \begin{cases} 1 & \text{if } s' \text{ is even} \\ 2 & \text{if } s' \text{ is odd and } w = 1 \\ 0 & \text{if } s' \text{ is odd and } w = -1. \end{cases}$$

PROOF. Suppose we choose a in equation (3.2.9) to reduce to the central vertex v_0 of X. Then the definition of W_Ψ in (3.3.18) and Lemma 3.3.23 yield

$$W_\Psi^\pm = \sum_{e \in (v_0 \to \gamma_\Psi v_0)} \Phi_f^\pm \left([\infty] - \left[\frac{-\nu}{c} \right] \right) (e, P_\Psi).$$

The edges e in the sum may be written $e_j = \gamma_j^{-1} e_0$ where $\gamma_j = \begin{pmatrix} 1 & -\nu' \\ 0 & p^j \end{pmatrix}$, where ν' is an integer such that $\nu' \equiv -\nu/c \pmod{p^{2s}}$, and $j = 0, \ldots, 2s-1$. We evaluate each term in the sum:

$$\Phi_f^\pm \left([\infty] - \left[-\frac{\nu}{c} \right] \right) (\gamma_j^{-1} e_0, P_\Psi) = w^{|\gamma_j|} \Phi_f^\pm \left([\gamma_j \infty] - \left[\gamma_j \left(-\frac{\nu}{c} \right) \right] \right) (e_0, P_\Psi|_{\gamma_j^{-1}})$$

$$= w^j \Phi_f^\pm \left([\infty] - \left[\frac{(-\nu - c\nu')/p^j}{c} \right] \right) \left(e_0, \left(cz + \frac{c\nu' + \nu}{p^j} \right)^{\frac{k-2}{2}} \right). \tag{3.3.25}$$

From the invariance of f under the transformation $z \mapsto z+1$, it is clear that the expression in (3.3.25) depends on the integer $(c\nu' + \nu)/p^j$ only up to its equivalence class modulo c. As $j = 0, \ldots, 2s-1$, these integers run over the set J_ν: once if s' is even and twice if s' is odd. In the latter case, the coefficients w^j appear with opposite sign in the two occurences when $w = -1$, and with the same sign when $w = 1$. The result follows. □

We may now prove the first half of theorem 3.3.19. Let χ be a Dirichlet character of conductor c with $\chi(p) = w$ and $\chi(-1) = w_\infty$. Note that $\beta \neq 0$, and hence $s' = 2s/\beta$. Then by Proposition 3.3.24 we have:

$$\frac{1}{2s} \sum_{\nu \in (\mathbf{Z}/c\mathbf{Z})^\times} \chi(\nu) W_\Psi^{w_\infty} =$$

$$\frac{1}{s'} \sum_{\nu \in (\mathbf{Z}/c\mathbf{Z})^\times} \chi(\nu) \sum_{a \in J_\nu} w^{j(a)} \phi_f^{w_\infty} \left(\left[-\frac{a}{c} \right] - [\infty] \right) \left((cz+a)^{(k-2)/2} \right). \tag{3.3.26}$$

As ν ranges over $(\mathbf{Z}/c\mathbf{Z})^\times$, the sets J_ν cover $(\mathbf{Z}/c\mathbf{Z})^\times$ with each element repeated s' times. Furthermore, for $a \in J_\nu$ we have $\chi(\nu) w^{j(a)} = \chi(\nu \cdot p^{j(a)}) = \chi(a)$. We find that (3.3.26) equals

$$\sum_{a \in (\mathbf{Z}/c\mathbf{Z})^\times} \chi(a) \phi_f^{w_\infty} \left(\left[-\frac{a}{c} \right] - [\infty] \right) \left((cz+a)^{(k-2)/2} \right).$$

Equation (3.3.20) now follows from Lemma 3.2.3.

3.3.4. *Second part of the Proof of Orton's Theorem.* In this section, we relate the distribution λ_f to the p-adic L-function of f. Let us now compare the distribution $\mu^{\pm}_{f,MTT}$ on $\mathbf{Z}^{\times}_{p,c}$ to the Darmon-Orton modular symbol of distributions $\lambda^{\pm}_f$. Let Ψ be an embedding as in section 3.3.2. For each integer i, define

$$U(v_i) := \{t \in \mathbf{P}^1(\mathbf{Q}_p) - \{x_\Psi, y_\Psi\} : \mathrm{ord}(M_\Psi(t)) = i\}.$$

The motivation for this notation is as follows. Let $(x_\Psi \to y_\Psi)$ denote the bi-infinite path from the end of X corresponding to x_Ψ to the end corresponding to y_Ψ. The vertices of $(x_\Psi \to y_\Psi)$ may be labelled $\{v_i\}$ in such a way that $U(v_i)$ is the set of points corresponding to ends of X that intersect $(x_\Psi \to y_\Psi)$ precisely at v_i. If e_i is the edge from v_{i-1} to v_i, then $U(v_i) = U(e_i) - U(e_{i+1})$. A fundamental region for the action of γ_Ψ on $\mathbf{P}^1(\mathbf{Q}_p) - \{x_\Psi, y_\Psi\}$ is given by:

$$\mathcal{F}_\Psi := \bigcup_{i=0}^{2s-1} U(v_i).$$

For $z \in \mathcal{F}_\Psi$, write $i(z) = \mathrm{ord}(M_\Psi(z))$, i.e. the index i such that $z \in U(v_i)$. We also define

$$J_{\infty,\nu} = \{a \in \mathbf{Z}^{\times}_{p,c} : a \equiv \nu p^j \pmod{c} \text{ for some } j = j(a)\}.$$

Proposition 3.3.27. *If F is a locally analytic function on $\mathbf{Z}^{\times}_p$, then*

$$\int_{F_\Psi} p^{i(z)\cdot\frac{k-2}{2}} F\left(\frac{cz+\nu}{p^{i(z)}}\right) d\lambda^{\pm}_f([x_\Psi] - [y_\Psi])(z) = \beta \int_{J_{\infty,\nu}} w^{j(x)} F(x_p) d\mu^{\pm}_{f,\mathrm{MTT}}(x).$$

Proof. For $j = 0, \dots, 2s-1$, write $J_{\infty,\nu,j} = \{a \in \mathbf{Z}^{\times}_{p,c} : b \equiv \nu p^j \pmod{c}\}$. We will show

(3.3.28)
$$w^j p^{j\cdot\frac{k-2}{2}} \int_{U(v_j)} F\left(\frac{cz+\nu}{p^j}\right) d\lambda^{\pm}_f([x_\Psi] - [y_\Psi])(z) = \int_{J_{\infty,\nu,j}} F(x_p) d\mu^{\pm}_{f,\mathrm{MTT}}(x).$$

The result will then follow by summing from $j = 0$ to $j = 2s-1$; as j varies the $J_{\infty,\nu,j}$ cover $J_{\infty,\nu}$ once if s' is even, twice if s' is odd, and with opposite sign in the latter case when $w = -1$.

To prove (3.3.28), fix an integer $n > 2s$. Refine each $U(v_j)$ by

$$U(v_j) = \bigcup_{a \in (\mathbf{Z}/p^n\mathbf{Z})} U_{j,a}, \quad \text{where} \quad U_{j,a} = \left\{t \in U(v_j) : (ct+\nu)/p^j \equiv a \pmod{p^n}\right\}$$

and correspondingly refine $J_{\infty,\nu,j}$ as

$$J_{\infty,\nu,j} = \bigcup_{a \in (\mathbf{Z}/p^n\mathbf{Z})^\times} D(b_{a,j}, n)$$

where $b_{a,j} = (\nu + c\nu')/p^j + ac$ and $\nu' \in \mathbf{Z}$ satisfies $\nu' \equiv -\nu/c \pmod{n+2s}$. Then from the definition of the distribution $\mu_{f,\mathrm{MTT}}$, we have for a polynomial P of degree $\leq k-2$:

$$\begin{aligned}\int_{D(b,n)} P(x) d\mu^{\pm}_{f,\mathrm{MTT}}(x) &= w^n p^{-n\left(\frac{k-2}{2}\right)} \phi^{\pm}_f\left([\infty] - \left[\frac{b}{p^n c}\right]\right)(P(p^n cz + b)) \\ &= w^n p^{-n\left(\frac{k-2}{2}\right)} \Phi^{\pm}_f\left([\infty] - \left[\frac{b}{p^n c}\right]\right)(e_0, P(p^n cz + b)).\end{aligned} \tag{3.3.29}$$

where $b = b_{a,j}$. Now if $\gamma = \begin{pmatrix} 1 & -\nu' - p^j a \\ 0 & p^{n+j} \end{pmatrix}$, then $U_{j,a} = \gamma^{-1} U(e_0)$. Thus using the transformation property of $\Phi_f^{\pm}$ under γ, the right side of (3.3.29) may be written:

$$\begin{aligned} & w^j p^{-n\left(\frac{k-2}{2}\right)} \Phi_f^{\pm}\left(\left[\gamma^{-1}(\infty)\right] - \left[\gamma^{-1}\left(\frac{b}{p^n c}\right)\right]\right)\left(\gamma^{-1} e_0, P\left(p^n cz + b\right)|_{\gamma}\right) \\ = \quad & w^j p^{j\left(\frac{k-2}{2}\right)} \Phi_f^{\pm}\left([\infty] - \left[\frac{\nu}{c}\right]\right)\left(\gamma^{-1} e_0, P\left(\frac{cz+\nu}{p^j}\right)\right) \\ = \quad & w^j p^{j\left(\frac{k-2}{2}\right)} \int_{U_{j,a}} P\left(\frac{cz+\nu}{p^j}\right) d\lambda_f^{\pm}([x_\Psi] - [y_\Psi])(z) \end{aligned}$$

Thus we have proven the result for polynomials of degree $\leq k-2$ on the arbitrarily small balls $D(b,n)$ and $U(j,a)$. By the uniqueness properties of the extensions of $\mu^{\pm}_{f,\mathrm{MTT}}$ and $\lambda_f^{\pm}$ from the distributions on P_{k-2} to the space of locally analytic functions, the result follows. $\square$

3.3.5. *End of Orton's Theorem.* In this section we conclude the proof of Theorem 3.3.19. Recall the definition:

$$LI_\Psi^{\pm} = \int_{\mathbf{P}^1(\mathbf{Q}_p)} \log\left(\frac{x - \gamma_\Psi z}{x - z}\right) P_\Psi(x) d\lambda_f^{\pm}([x_\Psi] - [y_\Psi])(x).$$

Recall also how $\lambda_f^{\pm}([x_\Psi] - [y_\Psi])$ is applied to a locally analytic function such as $\log(\frac{x-\gamma_\Psi z}{x-z}) P_\Psi(x)$: we cover $\mathbf{P}^1(\mathbf{Q}_p)$ by smaller and smaller open balls, write the function as a power series on each open ball, truncate the power series to a polynomial of degree $k-2$, evaluate $\lambda_f^{\pm}([x_\Psi] - [y_\Psi])$ on each of these polynomials on the open balls via $\Phi_f^{\pm}([x_\Psi] - [y_\Psi])$, and sum the results; the limit as the covers become uniformly finer is the desired value.

In the present case, we write

$$\mathbf{P}^1(\mathbf{Q}_p) = \gamma_\Psi^{-n} U(\overline{e}_0) \sqcup \bigsqcup_{j=-n}^{n} \gamma_\Psi^j \mathcal{F}_\Psi \sqcup \gamma_\Psi^{n+1} U(e_0), \tag{3.3.30}$$

where $\overline{e}_0$ denotes the edge e_0 with the opposite orientation. We will refine the middle term of (3.3.30) later, but indicate first why, in the limit, the end divisions contribute nothing to the integral. Let $T_n(x)$ denote the truncation of the power series of $\log(\frac{x-\gamma_\Psi z}{x-z}) P_\Psi(x)$ expanded around y_Ψ on the open set $\gamma_\Psi^{n+1} U(e_0)$ to a polynomial of degree $k-2$. From the invariance of Φ under Γ, and the fact that γ_Ψ stabilizes x_Ψ, y_Ψ, and P_Ψ, we have

$$\begin{aligned} & \lim_{n\to\infty} \int_{\gamma^{n+1} U(e_0)} \log\left(\frac{x - \gamma_\Psi z}{x - z}\right) P_\Psi(x) d\lambda_f^{\pm}([x_\Psi] - [y_\Psi])(x) \\ = \quad & \lim_{n\to\infty} \Phi_f^{\pm}([x_\Psi] - [y_\Psi])(\gamma^{n+1} e_0, T_n(x)) \\ = \quad & \lim_{n\to\infty} \Phi_f^{\pm}([x_\Psi] - [y_\Psi])(e_0, V_n(x)), \end{aligned}$$

where V_n is the truncation to a polynomial of degree $k-2$ of the power series of

$$\log\left(\frac{\gamma_\Psi^{n+1} x - \gamma_\Psi z}{\gamma_\Psi^{n+1} x - z}\right) P_\Psi(x) = \log\left(\frac{x - \gamma_\Psi^{-n} z}{x - \gamma_\Psi^{-(n+1)} z}\right) P_\Psi(x)$$

on $U(e_0)$ expanded around y_Ψ. We leave it to the reader to verify (or consult [**28**, §6.4]) the explicit formula:

$$V_n(x) = \sum_{i=1}^{\frac{k-2}{2}} \left(-\left(\gamma_\Psi^{-n} z - y_\Psi\right)^{-i} + \left(\gamma_\Psi^{-(n+1)} z - y_\Psi\right)^{-i}\right) \frac{c^{\frac{k-2}{2}}}{i} (x - y_\Psi)^{i + \frac{k-2}{2}}.$$

As $n \to \infty$, $\gamma_\Psi^{-n} z \to x_\Psi = \infty$, so the coefficients above tend to zero. Thus we have proven that

$$\lim_{n\to\infty} \int_{\gamma^{n+1} U(e_0)} \log\left(\frac{x - \gamma_\Psi z}{x - z}\right) P_\Psi(x) d\lambda_f^\pm([x_\Psi] - [y_\Psi])(x) = 0,$$

and a similar result holds for the other end term in the decomposition (3.3.30).

Now we are left to analyze

$$LI_{\Psi,n}^\pm := \sum_{j=-n}^{n} \int_{\gamma_\Psi^j \mathcal{F}_\Psi} \log\left(\frac{x - \gamma_\Psi z}{x - z}\right) P_\Psi(t) d\lambda_f^\pm([x_\Psi] - [y_\Psi])(x).$$

For each term in the sum we invoke the change of variables $x \mapsto \gamma_\Psi^{-j} x$; using the $\Gamma_0^p(N)$-invariance of $\lambda_f^\pm$ we obtain

$$\begin{aligned} LI_{\Psi,n}^\pm &= \sum_{j=-n}^{n} \int_{\mathcal{F}_\Psi} \log\left(\frac{\gamma_\Psi^j x - \gamma_\Psi z}{\gamma_\Psi^j x - z}\right) P_\Psi(t) d\lambda_f^\pm([x_\Psi] - [y_\Psi])(x) \\ &= \sum_{j=-n}^{n} \int_{\mathcal{F}_\Psi} \log\left(\frac{x - \gamma_\Psi^{1-j} z}{x - \gamma_\Psi^{-j} z}\right) P_\Psi(t) d\lambda_f^\pm([x_\Psi] - [y_\Psi])(x) \\ &= \int_{\mathcal{F}_\Psi} \log\left(\frac{x - \gamma_\Psi^{1+n} z}{x - \gamma_\Psi^{-n} z}\right) P_\Psi(t) d\lambda_f^\pm([x_\Psi] - [y_\Psi])(x), \end{aligned} \tag{3.3.31}$$

as the sum telescopes. Now in the limit as $n \to \infty$, we have $\gamma_\Psi^{1+n} z \to y_\Psi$ and $\gamma_\Psi^{-n} z \to x_\Psi$; thus we would like to say that in the limit, we can replace the argument of log in (3.3.31) by a linear fractional transformation taking y_Ψ to 0 and x_Ψ to ∞, namely, $M_\Psi(t) := t + \nu/c$. More precisely, let

$$M_n(t) = \frac{-y_\Psi \gamma_\Psi^{-n} z}{\gamma_\Psi^{n+1} z} \cdot \frac{x - \gamma_\Psi^{1+n} z}{x - \gamma_\Psi^{-n} z}.$$

Now since

$$\begin{aligned} &\int_{\mathcal{F}_\Psi} P_\Psi(x) d\lambda_f^\pm([x_\Psi] - [y_\Psi])(x) \\ = \;& \Phi_f^\pm([x_\Psi] - [y_\Psi])(e_{2s}, P_\Psi) - \Phi_f^\pm([x_\Psi] - [y_\Psi])(e_0, P_\Psi) \\ = \;& \Phi_f^\pm([x_\Psi] - [y_\Psi])(\gamma_\Psi e_0, P_\Psi) - \Phi_f^\pm([x_\Psi] - [y_\Psi])(e_0, P_\Psi) \\ = \;& 0, \end{aligned}$$

it follows that

$$LI_{\Psi,n}^\pm = \int_{\mathcal{F}_\Psi} \log(M_n(x)) P_\Psi(x) d\lambda_f^\pm([x_\Psi] - [y_\Psi])(x).$$

Since $M_n(x) \to M_\Psi(x)$ as $n \to \infty$ for any $x \in \mathcal{F}_\Psi$, the continuity of λ gives

$$\begin{aligned} LI_\Psi^{\pm} &= \int_{\mathcal{F}_\Psi} \log(M_\Psi(x)) P_\Psi(x) d\lambda_f^{\pm}([x_\Psi] - [y_\Psi])(x) \\ &= \int_{\mathcal{F}_\Psi} \log(cx+\nu)(cx+\nu)^{\frac{k-2}{2}} d\lambda_f^{\pm}([x_\Psi] - [y_\Psi])(x) \\ &= \int_{\mathcal{F}_\Psi} p^{i(z)\cdot\frac{k-2}{2}} \log\left(\frac{cx+\nu}{p^{i(z)}}\right)\left(\frac{cx+\nu}{p^{i(z)}}\right)^{\frac{k-2}{2}} d\lambda_f^{\pm}([x_\Psi] - [y_\Psi])(x) \\ &= \beta \int_{J_{\infty,\nu}} w^{j(t)} t_p^{\frac{k-2}{2}} \log(t_p) d\mu_{f,\mathrm{MTT}}^{\pm}(t), \end{aligned}$$

by Proposition 3.3.27. We are now in a position to conclude the proof of Theorem 3.3.19. Let χ be a Dirichlet character of conductor c with $\chi(p) = w$ and $\chi(-1) = w_\infty$. We evaluate:

$$\begin{aligned} \frac{1}{2s} \sum_{\nu \in (\mathbf{Z}/c\mathbf{Z})^\times} \chi(\nu) LI_\Psi^{w_\infty} &= \frac{1}{s'} \sum_{\nu \in (\mathbf{Z}/c\mathbf{Z})^\times} \chi(\nu) \int_{J_{\infty,\nu}} w^{j(t)} t_p^{\frac{k-2}{2}} \log(t_p) d\mu_{f,\mathrm{MTT}}^{w_\infty}(t) \\ &= \frac{1}{s'} \sum_{\nu \in (\mathbf{Z}/c\mathbf{Z})^\times} \int_{J_{\infty,\nu}} \chi(t) t_p^{\frac{k-2}{2}} \log(t_p) d\mu_{f,\mathrm{MTT}}^{w_\infty}(t). \end{aligned} \tag{3.3.32}$$

Now as ν ranges over $(\mathbf{Z}/c\mathbf{Z})^\times$, the sets $J_{\infty,\nu}$ cover $\mathbf{Z}_{p,c}^\times$, with each point being covered s' times. Thus (3.3.32) becomes

$$\begin{aligned} \int_{\mathbf{Z}_{p,c}^\times} \chi(t) t_p^{\frac{k-2}{2}} \log(t_p) d\mu_{f,\mathrm{MTT}}^{w_\infty} &= \int_{\mathbf{Z}_{p,c}^\times} \chi(t)(\omega(t)\langle t\rangle)^{\frac{k-2}{2}} \log(t_p) d\mu_{f,\mathrm{MTT}}^{w_\infty} \\ &= \frac{d}{ds}\left(\int_{\mathbf{Z}_{p,c}^\times} \chi(t)\omega(t)^{\frac{k-2}{2}} \langle t\rangle^s d\mu_{f,\mathrm{MTT}}^{w_\infty}\right)|_{s=\frac{k-2}{2}} \\ &= \frac{d}{ds} L_p^{w_\infty}(f, \omega^{\frac{k-2}{2}}\chi, s)|_{s=\frac{k-2}{2}}. \end{aligned}$$

Finally, we remark that

$$(\omega^{\frac{k-2}{2}}\chi)(-1) = w_\infty(-1)^{\frac{k-2}{2}},$$

so

$$L_p(f, \omega^{\frac{k-2}{2}}\chi, s) = L_p^{w_\infty}(f, \omega^{\frac{k-2}{2}}\chi, s).$$

This concludes the proof of Theorem 3.3.19.

4. Breuil duality and p-adic Langlands theory

4.1. Brief remarks on the p-adic Langlands program. In this lecture, we approach the Darmon-Orton theory developed earlier from the point of view of p-adic Langlands theory. In general terms, the classical Langlands program sets up a correspondence between Galois representations and classical automorphic forms. The p-adic Langlands program, which is currently in an early but exciting phase of development, seeks (at least in its local version) to relate classes of continuous p-adic representations of reductive groups to local p-adic Galois representations. Because p-adic representations are so much more complicated than complex representations, and continuous p-adic representations are more complicated than classical smooth representations, the p-adic Langlands program requires the introduction of many new concepts. For an overview of some of the ideas in the (local) p-adic Langlands program, see the introduction to Breuil's paper [**2**]. See also the paper [**14**], which adopts a representation theoretic perspective on many of the ideas we have discussed.

To get a taste of these new ideas in the situation of interest to us in these lectures, let us denote by $\sigma(f)$ the p-adic representation of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ attached to the newform f of weight $k \geq 2$ and level M. The philosophy of the p-adic Langlands program suggests that one should be able to recover this Galois representation from purely automorphic data associated to f. Similarly, the "local" p-adic Langlands program should relate the local representation $\sigma_p(f) := \sigma(f)|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)}$ to the local automorphic component $\pi_p(f)$ of the representation of GL_2 determined by f.

From the classical theory, we know that when the level of f is exactly divisible by p, the local automorphic component $\pi_p(f)$ does not contain enough information to isolate σ_p. Indeed, in this case, $\pi_p(f)$ is always the Steinberg representation and it is exactly the $\mathcal{L}$ invariant of f (and its weight) that provide the additional information necessary to identify the local Galois representation associated to f. In the papers [**2**] and [**3**] Breuil seeks to answer the question:

> How can we extract the $\mathcal{L}$-invariant of f from "automorphic" information?

Breuil's answer to this question begins with a certain p-adic completion $\hat{H}^1_c(N) \otimes E$ of the étale cohomology of the tower of modular curves of level Np^r. This completion is a p-adic Banach space with a Hecke action and $\mathrm{GL}_2(\mathbf{Q}_p)$ action that preserves its norm, defined over the finite extension E of $\mathbf{Q}_p$ generated by the Hecke eigenvalues of f. (This space is defined in more detail in the next section.) Roughly speaking, the part of the space $\hat{H}^1_c(N) \otimes E$ cut out by insisting that the Hecke algebra act through the eigenvalues of the form f contains the locally algebraic representation

$$\mathrm{Sym}^{k-2}(E^2) \otimes_E \pi_p(f) \text{ ``}\cong\text{'' } C^{la}(\mathbf{Q}_p, 2-k)/P_{k-2} \otimes E$$

embedded $\mathrm{GL}_2(\mathbf{Q}_p)$–equivariantly, where the symbol "$\cong$" means "more or less the same as." For a thorough discussion of these ideas, see [**14**, Section 4], which in turn refers to [**13**].

Denote by $\hat{\pi}_p(f)$ the closure of $\mathrm{Sym}^{k-2} E^2 \otimes_E \pi_p(f)$ in $\hat{H}^1_c(N) \otimes E$. Breuil proves that:

(1) when $k > 2$, $\sigma_p(f)$ is absolutely irreducible, and $\hat{\pi}_p(f)$ indeed exactly determines $\mathcal{L}(f)$.
(2) When $k = 2$, $\sigma_p(f)$ is reducible and is not determined by $\hat{\pi}_p(f)$. However, Breuil shows that $\hat{H}^1_c(N) \otimes E$ contains a topologically reducible Banach space representation of length 2 with $\hat{\pi}_p(f)$ its unique sub-object, which determines $\mathcal{L}(f)$, and which depends only on $\sigma_p(f)$.

For the purposes of these notes, we will be mainly interested in how Breuil's point of view gives an alternate definition of the $\mathcal{L}$-invariant. As before, we assume that the level of f is $M = Np$, with $p \nmid N$. In this situation, the sign w occurring in the previous section is $w = a_p^{-1} \cdot p^{\frac{k-2}{2}}$. Let $\mathrm{nr}(w)$ denote the representation of $\mathrm{GL}_2(\mathbf{Q}_p)$ which sends $x \mapsto w^{\mathrm{ord}_p(\det(x))}$. For each $\mathcal{L} \in E$, Breuil uses the theory of modular symbol-valued measures on the upper half plane to define a Banach space representation $B(k, \mathcal{L})$ of $\mathrm{GL}_2(\mathbf{Q}_p)$. He then proves:

THEOREM 4.1.1. *There exists a (unique) $\mathcal{L}_B(f) \in E$ such that*

$$\mathrm{Hom}_{\mathrm{GL}_2(\mathbf{Q}_p)}\left(B(k,\mathcal{L}) \otimes \mathrm{nr}(w), (\hat{H}^1_c(N) \otimes E)^f\right) \cong \begin{cases} 0 & \text{if } \mathcal{L} \neq -\mathcal{L}_B(f) \\ E^2 & \text{if } \mathcal{L} = -\mathcal{L}_B(f). \end{cases}$$

Furthermore, we have $\mathcal{L}_B(f) = \mathcal{L}_O^+(f) = \mathcal{L}_O^-(f)$.

Here $(\hat{H}^1_c(N) \otimes E)^f$ denotes the f-isotypic component of $\hat{H}^1_c(N) \otimes E$, i.e. the subspace on which the Hecke algebra acts via the eigenvalues of f. In the remainder of this lecture, we discuss $B(k,\mathcal{L})$ and sketch a proof of Theorem 4.1.1.

4.2. Completed Étale cohomology. Let $K = \prod_\ell K_\ell$ be an open compact subgroup of $\mathrm{GL}_2(\hat{\mathbf{Z}})$. We denote by $Y(K)$ the open modular curve over $\mathbf{Q}$, whose complex points are given by:

$$Y(K)(\mathbf{C}) := \mathrm{GL}_2(\mathbf{Q})\backslash \mathrm{GL}_2(\mathbf{A}_\mathbf{Q})/\,\mathrm{SO}_2(\mathbf{R})\mathbf{R}^\times K.$$

The number of geometric connected components of $Y(K)$ is the size of

$$(\mathbf{Q}^+)^\times\backslash\mathbf{A}^\times_{\mathbf{Q},f}/\det(K).$$

For a positive integer n, we write $H^1(Y(K), \mathbf{Z}/p^n\mathbf{Z})$ for the Betti cohomology of the complex space $Y(K)(\mathbf{C})$, or equivalently, the étale cohomology of the algebraic variety $Y(K)_{\overline{\mathbf{Q}}}$. Similarly, $H^1_c(Y(K), \mathbf{Z}/p^n\mathbf{Z})$ represents cohomology with compact supports. For $* = c$ or $* =$ empty, the $\mathbf{Z}/p^n\mathbf{Z}$-module $H^1_*(Y(K), \mathbf{Z}/p^n\mathbf{Z})$ is naturally endowed with an action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. In the most concrete terms, the Galois action on $H^1_c(Y(K), \mathbf{Z}/p^n\mathbf{Z})$ can be understood by identifying this space with the p^n torsion of the jacobian of the closed curve $X(K)$, relative to its set of cusps.

In addition, $H^1_*(Y(K), \mathbf{Z}/p^n\mathbf{Z})$ is endowed with a Hecke action. First let

$$H^1_*(Y(K), \mathbf{Z}_p) = \varprojlim H^1_*(Y(K), \mathbf{Z}/p^n\mathbf{Z}).$$

Note that for $K' \subset K$, the group K acts on the right on $Y(K')$ and hence on the cohomology $H^1_*(Y(K'), \mathbf{Z}_p)$. Furthermore, the collection of such K' forms a direct system under inclusion, with an inclusion $K_1 \subset K_2 \subset K$ inducing a K-equivariant map

$$H^1_*(Y(K_2), \mathbf{Z}_p) \to H^1_*(Y(K_1), \mathbf{Z}_p).$$

If we take the direct limit of this system, it is clear that $H^1_*(Y(K), \mathbf{Z}_p)$ maps to this direct limit, and lies in the subspace which is invariant under the action of K:

$$H^1_*(Y(K), \mathbf{Z}_p) \subset (\varinjlim_{K'} H^1_*(Y(K'), \mathbf{Z}_p))^K.$$

Now we may define the Hecke action. For each ℓ with $K_\ell = \mathrm{GL}_2(\mathbf{Z}_\ell)$, write

$$K\begin{pmatrix}1 & 0\\ 0 & \ell\end{pmatrix}K = \bigsqcup \delta_i K$$

for the matrix

$$\begin{pmatrix}1 & 0\\ 0 & \ell\end{pmatrix} \in \mathrm{GL}_2(\mathbf{Q}_\ell) \subset \mathrm{GL}_2(\mathbf{A}_\mathbf{Q}).$$

For $x \in H^1_*(Y(K), \mathbf{Z}_p)$, define $T_\ell(x) := \sum \delta_i^{-1}x$ in $(\lim_{\rightarrow K'} H^1_*(Y(K'), \mathbf{Z}_p))^K$; one must check that the image $T_\ell(x)$ again lies in $H^1_*(Y(K), \mathbf{Z}_p)$. This action induces the action on the quotients $H^1_*(Y(K), \mathbf{Z}/p^n\mathbf{Z}) = H^1_*(Y(K), \mathbf{Z}_p)/p^n$ as well. The

Hecke operator S_ℓ is defined similarly via the matrix $\begin{pmatrix} \ell & 0 \\ 0 & \ell \end{pmatrix}$. The Galois and Hecke actions commute.

Now let $K^p = \prod_{\ell \neq p} K_\ell$ be an open compact subgroup of $\mathrm{GL}_2(\hat{\mathbf{Z}})$, with trivial component at p. Let $K(p^r) := \ker(\mathrm{GL}_2(\mathbf{Z}_p) \to \mathrm{GL}_2(\mathbf{Z}/p^r\mathbf{Z}))$ and define

$$H^1_*(K^p) := \lim_{\to r} H^1_*(Y(K^pK(p^r)), \mathbf{Z}_p)$$

The transition maps in the inductive limit for $*$ = empty are the usual contravariant maps on cohomology induced by the projection maps $Y(K^pK(p^{r+1}) \to Y(K^pK(p^r))$. For $* = c$, the transition maps are the duals of the trace maps $H^1_*(Y(K^pK(p^{r+1})), \mathbf{Z}_p) \to H^1_*(Y(K^pK(p^r)), \mathbf{Z}_p)$ induced by the projections. The $\mathbf{Z}_p$-module $H^1_*(K^p)$ is torsion-free. Furthermore, it is endowed with a smooth left $\mathrm{GL}_2(\mathbf{Q}_p)$-action which we now describe.

For each $\gamma \in \mathrm{GL}_2(\mathbf{Q}_p)$, right multiplication by γ^{-1} induces a map $Y(\gamma^{-1}K\gamma) \mapsto Y(K)$, which in turn gives a map $H^1_*(Y(K), \mathbf{Z}_p) \to H^1_*(Y(\gamma^{-1}K\gamma), \mathbf{Z}_p)$. Given any element $x \in H^1_*(K^p)$, we may choose an index r large enough so that x is represented by an element in $H^1_*(K^pK(p^r), \mathbf{Z}_p)$, and such that $\gamma^{-1}K^pK(p^r)\gamma \subset K^pK(p^s)$ for some $s > 1$. Then the image of x under the composition of maps

$$H^1_*(K^pK(p^r), \mathbf{Z}_p) \to H^1_*(\gamma^{-1}K^pK(p^r)\gamma, \mathbf{Z}_p) \to H^1_*(K^pK(p^s), \mathbf{Z}_p)$$

represents an element of $H^1_*(K^p)$, which is defined to be γx.

Finally, we define

$$\hat{H}^1_*(K^p) := \lim_{\leftarrow n} \left(\lim_{\to r} H^1_*(Y(K^pK(p^r)), \mathbf{Z}/p^n\mathbf{Z}) \right) \cong \lim_{\leftarrow n} H^1_*(K^p)/p^n.$$

For the remainder of this lecture, we will be interested in particular in the case

$$K^p = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2\left(\prod_{\ell \neq p} \mathbf{Z}_\ell \right) : c \equiv 0 \pmod{N},\ a \equiv 1 \pmod{N} \right\}.$$

In this case, $Y(K^pK(p^r)) = Y(N, p^r) := Y_1(N) \times_{Y(1)} Y(p^r)$, the open modular curve whose connected geometric component $Y^0(N, p^r)$ can be identified with $\Gamma_1(N) \cap \Gamma(p^r) \backslash \mathcal{H}$.

The modules $H^1_*(K^p)$ and $\hat{H}^1(K^p)$ are endowed with $\mathbf{Z}_p$-linear actions of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, the Hecke operators T_ℓ, S_ℓ for $\ell \nmid Np$, and $\mathrm{GL}_2(\mathbf{Q}_p)$. This last action endows $\hat{H}^1_c(K^p) \otimes \mathbf{Q}_p$ with the structure of an admissible unitary Banach space representation of $\mathrm{GL}_2(\mathbf{Q}_p)$. We will simply write $H^1_c = H^1_c(N)$ and $\hat{H}^1_c = \hat{H}^1_c(N)$ for $H^1_c(K^p)$ and $\hat{H}^1_c(K^p)$, respectively.

4.3. $\mathrm{GL}_2(\mathbf{Q}_p)$ representations and modular symbols. In order to connect the theory of p-adic Banach representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ with the work of Orton, Breuil gave a reinterpretation of the space of $\mathrm{GL}_2(\mathbf{Q}_p)$-equivariant maps from an arbitrary Banach space into $\hat{H}^1_c$, in terms of modular symbols. Let E denote a finite extension of $\mathbf{Q}_p$.

THEOREM 4.3.1 ([**2**], Théorème 2.4.2). *Let B be a unitary p-adic Banach space representation of $\mathrm{GL}_2(\mathbf{Q}_p)$ with coefficients in E, and let $B^* := \mathrm{Hom}_E(B, E)$ be its $\mathrm{GL}_2(\mathbf{Q}_p)$ unitary Banach dual. We have a canonical Hecke equivariant isomorphism*

$$\mathrm{Hom}_{\mathrm{GL}_2(\mathbf{Q}_p)}(B, \hat{H}^1_c \otimes_{\mathbf{Z}_p} E) \cong \mathrm{Hom}_{\tilde{\Gamma}^p_1(N)}(D_0, B^*), \tag{4.3.2}$$

where the left hand side denotes continuous E-linear $\mathrm{GL}_2(\mathbf{Q}_p)$-equivariant maps between the two indicated unitary Banach space representations, and the right side denotes B^*-valued modular symbols ϕ such that

$$\phi([\gamma r_1]-[\gamma r_2])(b)=\phi([r_1]-[r_2])(\gamma^{-1}b)$$

for all $\gamma\in\tilde{\Gamma}_1^p(N)$. (Note: a unitary representation means that the norm is G-invariant).

We state this theorem as Breuil does, for the groups

$$\tilde{\Gamma}_1^p(N)=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in \mathrm{GL}_2(\mathbf{Z}[1/p])^+ :\ c\equiv 0 \pmod N,\ a\equiv 1 \pmod N\right\}$$

and

$$\Gamma_1^p(N)=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in \mathrm{SL}_2(\mathbf{Z}[1/p])^+ :\ c\equiv 0 \pmod N,\ a\equiv 1 \pmod N\right\}.$$

This is a slightly more general setting then that in the previous section, where we worked with Γ_0^p. No doubt the results in the previous section hold in this more generally setting, since the difference between Γ_1^p and Γ_0^p is "away from p," but we have not attempted to make this generalization and we content ourselves here with following Breuil.

The Hecke equivariance in Thoerem 4.3.1 is with respect to the operators T_ℓ and S_ℓ for $\ell\nmid Np$, and w_∞. These operators act on the left of (4.3.2) via their action on $\hat{H}_c^1$, and on the right via the usual action on modular symbols.

To prove Theorem 4.3.1, let M be the closed unit ball in B. We will show that there is a Hecke equivariant isomorphism

(4.3.3)
$$\mathrm{Hom}_{\mathrm{GL}_2(\mathbf{Q}_p)}\left(M, H_c^1\otimes \mathcal{O}_E/\pi_E^n\mathcal{O}_E\right)\cong \mathrm{Hom}_{\tilde{\Gamma}_1^p(N)}\left(D_0,\mathrm{Hom}_{\mathcal{O}_E}(M,\mathcal{O}_E/\pi_E^n\mathcal{O}_E)\right).$$

Theorem 4.3.1 follows by passing to the limit over n and tensoring with E.

To prove (4.3.3), we begin by providing an alternate description of the left hand side. Denote by $\mathrm{Ind}_1^{\mathrm{GL}_2(\mathbf{Z}/p^r\mathbf{Z})} 1_{\mathbf{Z}/p^n\mathbf{Z}}$ the $\mathbf{Z}/p^n\mathbf{Z}$-module of functions $f:\mathrm{GL}_2(\mathbf{Z}/p^r\mathbf{Z})\to \mathbf{Z}/p^n\mathbf{Z}$. Denote by

$$\mathrm{Hom}_{\Gamma_1(N)}(D_0,\mathrm{Ind}_1^{\mathrm{GL}_2(\mathbf{Z}/p^r\mathbf{Z})} 1_{\mathbf{Z}/p^n\mathbf{Z}}) \tag{4.3.4}$$

the $\mathbf{Z}/p^n\mathbf{Z}$-module of group homomorphisms $\phi: D_0\to \mathrm{Ind}_1^{\mathrm{GL}_2(\mathbf{Z}/p^r\mathbf{Z})} 1_{\mathbf{Z}/p^n\mathbf{Z}}$ such that

$$\phi([r_1]-[r_2])(x\gamma)=\phi([\gamma r_1]-[\gamma r_2])(x)$$

for all $[r_1]-[r_2]\in D_0$, $x\in \mathrm{GL}_2(\mathbf{Z}/p^r\mathbf{Z})$, and $\gamma\in\Gamma_1(N)$. The $\mathbf{Z}/p^n\mathbf{Z}$-module in (4.3.4) is endowed with a left $\mathrm{GL}_2(\mathbf{Z}/p^r\mathbf{Z})$-module action via:

$$(g(\phi))([r_1]-[r_2])(x):=\phi([r_1]-[r_2])(g^{-1}x). \tag{4.3.5}$$

We then have:

LEMMA 4.3.6. *For each pair of integers $n>0$ and $r>1$, there is a canonical $\mathrm{GL}_2(\mathbf{Z}/p^r\mathbf{Z})$-equivariant isomorphism*

$$H_c^1(Y(N,p^r),\mathbf{Z}/p^n\mathbf{Z})\cong \mathrm{Hom}_{\Gamma_1(N)}(D_0,\mathrm{Ind}_1^{\mathrm{GL}_2(\mathbf{Z}/p^r\mathbf{Z})} 1_{\mathbf{Z}/p^n\mathbf{Z}}). \tag{4.3.7}$$

Furthermore, this isomorphism is equivariant with respect to the Hecke operators T_ℓ, S_ℓ for $\ell\nmid Np$, and transforms the action of complex conjugation on the left to that of w_∞ on the right.

PROOF. We will give the geometric intuition behind the slightly weaker isomorphism

$$H^1_c(Y^0(N,p^r),\mathbf{Z}/p^n\mathbf{Z}) \cong \mathrm{Hom}_{\Gamma_1(N)}(D_0,\mathrm{Ind}_1^{\mathrm{SL}_2(\mathbf{Z}/p^r\mathbf{Z})} 1_{\mathbf{Z}/p^n\mathbf{Z}}). \tag{4.3.8}$$

The curve $Y^0(N,p^r)$ may be thought of as the many holed torus $X^0(N,p^r)$, minus a certain finite number of cusps. By Poincaré duality, the left side of (4.3.8) is

$$\mathrm{Hom}(H_1(X^0(N,p^r),\mathrm{cusps};\mathbf{Z}),\mathbf{Z}/p^n\mathbf{Z}),$$

the $\mathbf{Z}/p^n\mathbf{Z}$-dual of the homology of the torus $X^0(N,p^r)$ relative to its set of cusps. Meanwhile, the right side of (4.3.8) is $\mathrm{Hom}_{\Gamma_1(N)\cap\Gamma(p^r)}(D_0,\mathbf{Z}/p^n\mathbf{Z})$ by Shapiro's Lemma. It thus remains to prove:

$$\mathrm{Hom}(H_1(X^0(N,p^r),\mathrm{cusps};\mathbf{Z}),\mathbf{Z}/p^n\mathbf{Z}) \cong \mathrm{Hom}_{\Gamma_1(N)\cap\Gamma(p^r)}(D_0,\mathbf{Z}/p^n\mathbf{Z}). \tag{4.3.9}$$

Given an element $[x]-[y]\in D_0$, consider any path in $\mathcal{H}\cup\mathbf{P}^1(\mathbf{Q})$ starting at x and ending at y. The image of this path in $\Gamma_1(N)\cap\Gamma(p^r)\backslash(\mathcal{H}\cup\mathbf{P}^1(\mathbf{Q}))$ yields a well-defined element of $H_1(X^0(N,p^r),\mathrm{cusps};\mathbf{Z})$. The theory of modular symbols states that this identification induces an isomorphism as in (4.3.9).

The compatibility of the isomorphism with the Hecke algebra involves computations with the group action. The isomorphism (4.3.7) follows from similar, but more involved reasoning. For details of all of this, see [**2**, Lemme 2.3.2]. □

Passing to the inductive limit over r in (4.3.7), we obtain a Hecke equivariant isomorphism

$$H^1_c/p^n \cong \mathrm{Hom}_{\Gamma_1(N)}(D_0,\mathrm{Ind}_1^{\mathrm{GL}_2(\mathbf{Z}_p)} 1_{\mathbf{Z}/p^n\mathbf{Z}}), \tag{4.3.10}$$

where $\mathrm{Ind}_1^{\mathrm{GL}_2(\mathbf{Z}_p)} 1_{\mathbf{Z}/p^n\mathbf{Z}}$ denotes the $\mathbf{Z}/p^n\mathbf{Z}$-module of locally constant functions $\mathrm{GL}_2(\mathbf{Z}_p)\to\mathbf{Z}/p^n\mathbf{Z}$. The right hand side of (4.3.10) is endowed with a left $\mathrm{GL}_2(\mathbf{Z}_p)$ action, as given in equation (4.3.5). This action may be extended to $\mathrm{GL}_2(\mathbf{Q}_p)$ as follows. For any $g\in\mathrm{GL}_2(\mathbf{Q}_p)$ and $x\in\mathrm{GL}_2(\mathbf{Z}_p)$, we may write $g^{-1}x=ba$ with $b\in\mathrm{GL}_2(\mathbf{Z}_p)$ and $a\in\tilde{\Gamma}^p_1(N)$. We then define

$$(g(\phi))([r_1]-[r_2])(x) := \phi([ar_1]-[ar_2])(b). \tag{4.3.11}$$

One easily checks that this definition is independent of the choice of a and b, and yields a well-defined $\mathrm{GL}_2(\mathbf{Q}_p)$-action extending the $\mathrm{GL}_2(\mathbf{Z}_p)$-action defined in (4.3.5). Furthermore, with this action, the isomorphism (4.3.10) is $\mathrm{GL}_2(\mathbf{Q}_p)$-equivariant.

We are now in a position to prove (4.3.3). In view of (4.3.10), we must construct an isomorphism

$$\begin{array}{c}\varphi : \mathrm{Hom}_{\mathrm{GL}_2(\mathbf{Q}_p)}\left(M,\mathrm{Hom}_{\Gamma_1(N)}(D_0,\mathrm{Ind}_1^{\mathrm{GL}_2(\mathbf{Z}_p)} 1_{\mathcal{O}_E/\pi_E^n\mathcal{O}_E})\right)\\ \big\downarrow \\ \mathrm{Hom}_{\tilde{\Gamma}^p_1(N)}\left(D_0,\mathrm{Hom}_{\mathcal{O}_E}(M,\mathcal{O}_E/\pi_E^n\mathcal{O}_E)\right).\end{array}$$

Such a map φ is given by

$$\varphi(F)([r_1]-[r_2])(m) := F(m)([r_1]-[r_2])(1)$$

for all $m\in M$. Its inverse ψ is given by

$$\psi(G)(m)([r_1]-[r_2])(h) := G([r_2]-[r_1])(h^{-1}m)$$

for all $m \in M$ and $h \in \mathrm{GL}_2(\mathbf{Z}_p)$. We leave it to the reader to verify (or consult [**3**, Proposition 2.4.1]) that the functions φ and ψ indeed map the indicated spaces to one another, that the two maps are mutually inverse, and that they are Hecke equivariant. This concludes the proof of Theorem 4.3.1.

4.4. Breuil Duality. Breuil Duality is a generalization of the Morita Duality that we discussed in Section 2.2.4. For details on this duality, see the proof of [**3**, Theorem 3.2.3]. In this section, we continue to assume that the ground field (over which $\mathcal{X}$ is defined) is the field $\mathbf{Q}_p$.

Let $\mathcal{O}_E(k)$ be the space of rigid analytic functions $H : \mathcal{X} \to E$ with a left $G = \mathrm{GL}_2(\mathbf{Q}_p)$ action given by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} H\right)(z) := \frac{\det(g)^{k/2}}{(bz+d)^k} H\left(\frac{az+c}{bz+d}\right).$$

Let $|\cdot|_p$ be the usual p-adic absolute value, and let ϵ be the character $\epsilon : \mathbf{Q}_p^* \to \mathbf{Z}_p^*$ defined by

$$\epsilon(x) = x|x|_p.$$

We view ϵ as a character of G through the determinant:

$$\epsilon(g) = \epsilon(\det(g)).$$

Let $\mathcal{O}_{E,\epsilon}(k)$ be the space obtained by adjusting the action of the center of G

$$\mathcal{O}_{E,\epsilon}(k) = \epsilon^{(2-k)/2} \otimes \mathcal{O}_E(k).$$

We now define the space $\mathcal{O}_E(k, \mathcal{L})$ for each $\mathcal{L} \in E$. This space consists of the integrals of functions in $\mathcal{O}_{E,\epsilon}(k)$. These integrals are not rigid analytic, but involve a fixed "branch" of the p-adic logarithm determined by the number $\mathcal{L}$. (See the partial fractions expansion in Lemma 2.2.7 and consider what's involved in integrating it formally).

Definition 4.4.1. Let $\log_{\mathcal{L}} : \mathbf{C}_p^\times \to \mathbf{C}_p$ be the branch of the p-adic logarithm which satisfies $\log_{\mathcal{L}}(p) = \mathcal{L}$.

Let

$$U = \bigsqcup_{i=0}^{s} U_i$$

be a covering of $\mathbf{Q}_p$ in $\mathbf{C}_p$ by pairwise disjoint opens U_i, such that $U_0 := \{z \in \mathbf{C}_p : |z| > r_0\}$ and $U_i := \{z \in \mathbf{C}_p : |z - z_i| < r_i\}$ for $1 \le i \le s$, with $r_i \in |E^\times|$ and $z_i \in \mathbf{Q}_p$. Define $\mathcal{O}(2-k, \mathcal{L})$ to be the space of functions $H : \mathcal{X} \to \mathbf{C}_p$ such that the restriction to each affinoid $\mathcal{X}_U := \mathbf{C}_p - U \subset \mathcal{X}$ with U as above, has the form:

$$H|_{\mathcal{X}_U} = H_U + \sum_{i=1}^{s} \sum_{n=0}^{k-2} c_{i,n} z^n \log_{\mathcal{L}}(z - z_i),$$

with $c_{i,n} \in E$ and H_U an E-rational rigid analytic function on $\mathcal{X}_U$. The space $\mathcal{O}(2-k, \mathcal{L}, E)$ is endowed with the left $\mathrm{GL}_2(\mathbf{Q}_p)$ action given by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} H\right)(z) := \epsilon(ad-bc)^{-\frac{k-2}{2}} \frac{(bz+d)^{k-2}}{(ad-bc)^{(k-2)/2}} H\left(\frac{az+c}{bz+d}\right).$$

As Lemma 2.2.7 suggests, the $(k-1)$st derivative map induces a short exact sequence of $\mathrm{GL}_2(\mathbf{Q}_p)$-representations:

$$0 \to P_{k-2}(E) \otimes \epsilon^{\frac{2-k}{2}} \to \mathcal{O}(2-k, \mathcal{L}, E) \to \mathcal{O}_{E,\epsilon}(k) \to 0. \tag{4.4.2}$$

Let $C_{\log}(\mathbf{Q}_p, 2-k, \mathcal{L}, E)$ be the space of locally analytic functions h on $\mathbf{Q}_p$ such that in a neighborhood of ∞ we have,

$$h(z) = -2P(z)\log_{\mathcal{L}}(z) + z^{k-2}\sum_{n=0}^{\infty}\frac{a_n}{z^n} \tag{4.4.3}$$

with $a_n \in E$ and $P(z) \in P_{k-2}(E)$. The space $C_{\log}(\mathbf{Q}_p, 2-k, \mathcal{L}, E)$ has a direct limit topology similar to that on $C^{an}(\mathbf{Q}_p, 2-k, E)$ defined in Definition 2.1.4. To be precise, given a covering U of $\mathbf{Q}_p$ in $\mathbf{C}_p$ as above, a collection of power series f_i on U_i, along with a function f_0 as in Equation (4.4.3) on the set U_0, gives an element f in $C_{\log}$. The functions defined relative to a fixed covering form a Banach space, and the full space $C_{\log}(\mathbf{Q}_p, 2-k, \mathcal{L}, E)$ is the direct limit of these Banach spaces as the coverings are refined.

The group action on $C_{\log}(\mathbf{Q}_p, 2-k, \mathcal{L}, E)$ is given by the formula

$$g_*(h)(z) = \epsilon^{(k-2)/2}(g)\frac{(bz+d)^{k-2}}{\det(g)^{(k-2)/2}}\left[h(\frac{az+c}{bz+d}) + P(\frac{az+c}{bz+d})\log_{\mathcal{L}}(\frac{ad-bc}{(bz+d)^2})\right].$$

Define

$$\Sigma(\mathbf{Q}_p, 2-k, \mathcal{L}, E) = C_{\log}(\mathbf{Q}_p, 2-k, \mathcal{L}, E)/P_{k-2}(E). \tag{4.4.4}$$

Let us now consider the dual exact sequence to (4.4.2). For even integer $k \geq 2$, Morita duality (Theorem 2.2.1) identifies the dual of the rightmost term $\mathcal{O}_{E,\epsilon}(k)$ in (4.4.2) with the space

$$C^{an}_{\epsilon}(\mathbf{Q}_p, 2-k, E)/P_{k-2} := \epsilon^{\frac{k-2}{2}} \otimes (C^{an}(\mathbf{Q}_p, 2-k, E)/P_{k-2}(E)),$$

where $C^{an}(\mathbf{Q}_p, 2-k, E)$ is the space of E-valued locally analytic functions on $\mathbf{Q}_p$ with poles of the correct order at ∞. Breuil duality (Theorem 4.4.5 below) identifies the dual of the larger space $\mathcal{O}(2-k, \mathcal{L}, E)$ with the space $\Sigma(\mathbf{Q}_p, 2-k, \mathcal{L}, E)$ of (4.4.4). Thus the dual exact sequence to (4.4.2) may be written:

$$\begin{tikzcd} 0 \arrow[r] & C^{an}_{\epsilon}(\mathbf{Q}_p, 2-k, E)/P_{k-2} \arrow[r] & \Sigma(\mathbf{Q}_p, 2-k, \mathcal{L}, E) \arrow[dl] \\ & \epsilon^{(k-2)/2} \otimes \mathrm{Hom}(P_{k-2}(E), E) \arrow[r] & 0. \end{tikzcd}$$

The map on the right side of this series picks out the logarithmic part at infinity of the function h, taking into account the group action.

Theorem 4.4.5. *There exists a unique G-invariant pairing*

$$\langle\ \cdot\ ,\ \cdot\ \rangle_B : C_{\log}(\mathbf{Q}_p, 2-k, \mathcal{L}, E)/P_{k-2}(E) \times \mathcal{O}(2-k, \mathcal{L}, E) \to E$$

satisfying:

(1) *for $z \in \mathcal{X}$, $\langle \frac{(x-z)^{k-2}}{(k-2)!}\log_{\mathcal{L}}(x-z), G\rangle = G(z)$;*
(2) *if $f \in C^{an}_{\epsilon}(\mathbf{Q}_p, 2-k, E)/P_{k-2}$ then*

$$\langle f, G\rangle_B = \langle f, G^{(k-1)}\rangle_M;$$

(3) *if* $G \in \mathcal{O}_{E,\epsilon}(2-k)$, *then*

$$\langle f, G\rangle_B = (-1)^{k-1}\langle f^{(k-1)}, G\rangle_M,$$

where $\langle\ \cdot\ ,\ \cdot\ \rangle_M$ *is the Morita pairing.*

PROOF. See [**2**, Section 3]. □

Of particular importance in Breuil's work are two Banach spaces, $B(2-k,\mathcal{L})$ and $B(2-k)$ that contain the spaces $\Sigma(\mathbf{Q}_p, 2-k, \mathcal{L}, E)$ and $C^{an}_{\epsilon}(\mathbf{Q}_p, 2-k, E)/P_{k-2}$ respectively as dense subspaces. One way to think of the space $B(2-k)$ is as the continuous dual of the space $\mathcal{O}_{\epsilon,E}(k)^b$ of functions $f \in \mathcal{O}_{\epsilon,E}(k)$ having bounded residues, defined as in Corollary 2.3.4 or Section 2.3 — and, in fact, this is literally true, provided one equips $\mathcal{O}_{\epsilon,E}(k)^b$ with the proper topology. It is also true that $B(2-k)$ is the completion of the space $C^{la}(\mathbf{Q}_p, 2-k, E)/P_{k-2}(E)$ of locally polynomial functions on $\mathbf{Q}_p$ with respect to a certain norm. When $k = 2$, this is just the sup-norm.

To avoid too many functional analytic complications, we content ourselves with listing some key properties of $B(2-k)$:

(1) $B(2-k)$ is a Banach space.
(2) When $k > 2$, it is the completion of the locally polynomial functions $C^{la}_{\epsilon}(\mathbf{Q}_p, 2-k, E)/P_{k-2}$ in a certain G-invariant norm.
(3) Let $C_{har}(k, \epsilon, E)$ denote the space of harmonic cocycles with values in $\mathrm{Hom}(P_{k-2}(E), E)$ and group action twisted by $\epsilon^{\frac{k-2}{2}}$. There is a continuous duality between the space $\mathcal{O}_{\epsilon,E}(k)^b \xrightarrow{\sim} C_{har}(k,\epsilon,E)^b$ of bounded rigid functions and $B(2-k)$; with the proper topology on the harmonic cocycles, each is the continuous dual of the other.
(4) The space $C^{an}_{\epsilon,E}(\mathbf{Q}_p, 2-k, E)/P_{k-2}$ is dense in $B(2-k)$.

Similarly, one can identify a space of bounded functions $\mathcal{O}(2-k,\mathcal{L},E)^b$ in the space $\mathcal{O}(2-k,\mathcal{L},E)$. In simple terms, a function is bounded if it, and all of its translates by the G-action, are bounded on a fixed affinoid domain in $\mathcal{X}$. This space of bounded functions is then given a topology so that its dual is a Banach space $B(2-k,\mathcal{L})$.

The relevant properties of $B(2-k,\mathcal{L})$ are:

(1) $B(2-k,\mathcal{L})$ is a Banach space with a G-invariant norm.
(2) There is a continuous duality between the "bounded" elements $\mathcal{O}(2-k,\mathcal{L},E)^b$ of $\mathcal{O}_{\epsilon,E}(2-k,\mathcal{L})$ and $B(2-k,\mathcal{L})$.
(3) When $k > 2$, the surjection $\mathcal{O}(\mathbf{Q}_p, 2-k,\mathcal{L}) \to \mathcal{O}_{\epsilon,E}(k)$ becomes an isomorphism.

To avoid too many functional-analytic complications, we will not give a precise definition of the Banach spaces $B(2-k)$ and $B(2-k,\mathcal{L})$. See [**3**, Section 3.1-3.3] for the full definition, which relies on the duality described in [**41**].

One important remark: the fact that $\mathcal{O}(2-k,\mathcal{L},E)^b$ is non-zero is highly nontrivial! The paper [**2**] proves this when $\mathcal{L} = \mathcal{L}(f)$ for some cusp for f of level Np; see the papers of Colmez for the general case.

4.5. Orton's $\mathcal{L}$-invariant from Breuil's viewpoint. In this section we combine the ideas of Breuil and Morita duality with the Darmon picture of modular symbols on the tree to approach Breuil's interpretation of the $\mathcal{L}$-invariant.

We first assemble a few facts. First, recall from Proposition 3.2.7 that, given a p-new cusp form f on $\Gamma_1(N)$ we have a function

$$\Phi_f^{\pm} : D_0 \times \mathrm{Edges}(X) \times P_{k-2} \to E$$

that, holding the D_0-variable fixed, is a bounded harmonic cocycle in $C_{har}(k)^b$ on X.

At the same time, the cohomological calculations from Section 3.3.1 tell us that

LEMMA 4.5.1. *We have* $w_\infty \Phi_f^{\pm} = \pm\Phi_f^{\pm}$, *and*

$$\mathrm{Hom}_{\Gamma_1^p(N)}(D_0, C_{har}(k))^f = E\Phi_f^+ \oplus E\Phi_f^-.$$

PROOF. This is essentially the content of Proposition 3.3.13. Since the map is Hecke equivariant, and the group $\Gamma_1^p(N)$ acts transitively on the edges (at least up to orientation) the function Φ_f is determined by its value on e_0, where it must lie in a one dimensional subspace. $\square$

THEOREM 4.5.2. *The residue map* $\mathcal{O}_{\epsilon,E}(k) \to C_{har}(k,\epsilon,E)$ *induces isomorphisms*

$$\begin{aligned}\mathrm{Hom}_{\Gamma_1^p(N)}(D_0, \mathcal{O}_{\epsilon,E}(k)^b) &\cong \mathrm{Hom}_{\Gamma_1^p(N)}(D_0, \mathcal{O}_{\epsilon,E}(k)) \\ &\cong \mathrm{Hom}_{\Gamma_1^p(N)}(D_0, C_{har}(k,\epsilon,E)^b).\end{aligned}$$

Furthermore,

$$\mathrm{Hom}_{\Gamma_1^p(N)}(D_0, \mathcal{O}(2-k,\mathcal{L})^b) \cong \mathrm{Hom}_{\Gamma_1^p(N)}(D_0, \mathcal{O}(2-k,\mathcal{L}))$$

In words: an invariant modular symbol with values in harmonic cocycles automatically takes values in bounded harmonic cocycles.

PROOF. We will prove the first result; the second is proved similarly, but since we avoided giving a precise definition of the norm on $\mathcal{O}(2-k,\mathcal{L})$, we won't give details. Suppose $\phi : D_0 \to \mathbf{C}_{har}(k,\epsilon,E)$ is $\Gamma_1^p(N)$ invariant. To compute the value $\phi(m)(e) \in \mathrm{Hom}(P_{k-2}(E), E)$, first use transitivity of the $\Gamma_1^p(N)$ action on the edges to find γ so that $e = \gamma e_0$. Then

$$\phi(m)(e) = \gamma(\phi(\gamma^{-1}(m))(e_0)).$$

Next use the fact that the stabilizer of e_0 in $\Gamma_1^p(N)$ is $\Gamma_1(pN)$ and that $D_0/\Gamma_1(pN)$ is finitely generated; in other words, there are finitely many $m_i \in D_0$, $\tau_i \in \Gamma_1(pN)$, and integers a_i so that

$$\gamma^{-1}(m) = \sum a_i \tau_i m_i.$$

Therefore

$$\omega(\gamma^{-1}\phi(m)(\gamma e_0)) = \omega(\phi(\gamma^{-1}(m))(e_0)) = \omega(\phi(\sum a_i\tau_i m_i)(e_0)) \le \inf_i \omega(\phi(m_i)(e_0)).$$

Thus the cocycle is automatically bounded. Now use the Poisson integral to integrate the associated measure (using Corollary 2.3.4) to obtain, for each $m \in D_0$, an element in $\mathcal{O}_{\epsilon,E}(k)$ which is, of necessity, bounded. $\square$

Let $\Phi_F^{\pm} \in \mathrm{Hom}_{\Gamma_1^p(N)}(D_0, \mathcal{O}(k))$ denote the elements which map to

$$\Phi_f^{\pm} \in \mathrm{Hom}_{\Gamma_1^p(N)}(D_0, C_{har}(k)^b)$$

under the isomorphism of Theorem 4.5.2. Using the Poisson integral, we may write

$$\Phi_F^{\pm}(m)(z) = \int_{\mathbf{P}^1(K)} \frac{1}{z-x} d\Phi_f^{\pm}(m)$$

viewing $\Phi_f^{\pm}$ as a bounded distribution.

Lemma 4.5.3. *We have $w_\infty \Phi_F^{\pm} = \pm\Phi_F^{\pm}$, and*

$$\mathrm{Hom}_{\Gamma_1^p(N)}(D_0, \mathcal{O}(k))^f = E\Phi_F^{+} \oplus E\Phi_F^{-}.$$

Proof. This follows directly from Lemma 4.5.1 and Theorem 4.5.2. □

Before we proceed, we introduce some extra notation. Fix $\mathcal{L} \in E$. If H is a rigid analytic E-rational function on Ω, we may view H as an element of $\mathcal{O}(2)$ and choose a lift $\tilde{H}$ of H via the surjection $\mathcal{O}(2, \mathcal{L}) \to \mathcal{O}(2)$; i.e. $\tilde{H}$ is a $\log_{\mathcal{L}}$-rigid antiderivative of H. The function $\tilde{H}$ is determined up to a constant, so for $z_1, z_2 \in \Omega$, the value

$$\int_{z_1}^{z_2} H(z)dz := \tilde{H}(z_1) - \tilde{H}(z_2)$$

is well-defined, and called a *Coleman line integral* relative to the choice of $\mathcal{L}$.

For each $\mathcal{L} \in E$ and $Q \in \Omega$, we define a 1-cocycle

$$c_{\mathcal{L},Q}^{\pm} \in Z^1(\Gamma_1^p(N), \mathrm{Hom}(D_0, \mathrm{Hom}(P_{k-2}(E), E)))$$

by the rule

$$\gamma \mapsto c_{\mathcal{L},Q}^{\pm}(\gamma)([r_1]-[r_2])(P(z)) := \int_Q^{\gamma Q} \Phi_F^{\pm}([r_1]-[r_2])(z)P(z)dz.$$

The class of $c_{\mathcal{L},Q}^{\pm}$ in $H^1(\Gamma_1^p(N), \mathrm{Hom}(D_0, \mathrm{Hom}(P_{k-2}(E), E)))$ is independent of Q and is denoted $c_{\mathcal{L}}^{\pm}$.

Proposition 4.5.4. *We have*

$$c_{\mathcal{L},Q}^{\pm}(\gamma)([r_1]-[r_2])(P(z)) = \Phi_f^{\pm}([r_1]-[r_2])\left(\log_{\mathcal{L}}\left(\frac{z-\gamma Q}{z-Q}\right)P(z)\right)$$

and hence

$$c_{\mathcal{L}}^{\pm} = lc_f^{\pm} + \mathcal{L}\cdot oc_f^{\pm}.$$

Proof. We give a completely formal, but essentially correct proof. Use the representation of Φ_F as a Poisson integral to write $c_{\mathcal{L},Q}^{\pm}$ as a "double integral":

$$c_{\mathcal{L},Q}^{\pm}(\gamma)([r_1]-[r_2])(P(z)) = \int_Q^{\gamma Q}\int_{\mathbf{P}^1(K)} \frac{P(z)}{z-x} d\Phi_f^{\pm}([r_1]-[r_2]).$$

Interchanging the order of integration, using the selected branch of the logarithm, and taking into account the fact that $d\Phi_f^{\pm}$ vanishes on polynomials, yields

$$\begin{aligned} c_{\mathcal{L},Q}^{\pm}(\gamma)([r_1]-[r_2])(P(z)) &= \int_{\mathbf{P}^1(\mathbf{Q}_p)}\int_Q^{\gamma Q} \frac{P(z)}{z-x} d\Phi_f^{\pm}([r_1]-[r_2]) \\ &= \int_{\mathbf{P}^1(\mathbf{Q}_p)} \log_{\mathcal{L}}\left(\frac{x-\gamma Q}{x-Q}\right) P(x) d\Phi_f^{\pm}([r_1]-[r_2]) \end{aligned}$$

as claimed. The last statement then follows from the fact that

$$\log_{\mathcal{L}}(z) = \log(z) + \mathcal{L}\,\mathrm{ord}(z).$$

□

THEOREM 4.5.5. *As usual, $k \geq 2$ and even. For every $\mathcal{L} \in E$, the surjection $\mathcal{O}(2-k,\mathcal{L}) \to \mathcal{O}(k)$ induces an injection:*

$$\mathrm{Hom}_{\Gamma_1^p(N)}(D_0, \mathcal{O}(2-k,\mathcal{L}))^f \hookrightarrow \mathrm{Hom}_{\Gamma_1^p(N)}(D_0, \mathcal{O}_{\epsilon,E}(k))^f, \tag{4.5.6}$$

and

$$\Phi_F^{\pm} \in \mathrm{Hom}_{\Gamma_1^p(N)}(D_0, \mathcal{O}(2-k,\mathcal{L}))^f \Leftrightarrow \mathcal{L} = \mathcal{L}_O^{\pm}(f).$$

PROOF. The exact sequence (4.4.2) induces a Hecke- and $\mathrm{GL}_2(\mathbf{Q})$-equivariant sequence

$$\begin{aligned} 0 \;\to\; & \mathrm{Hom}\left(D_0, P_{k-2}(E) \otimes \epsilon^{-\frac{k-2}{2}}\right) \to \mathrm{Hom}(D_0, \mathcal{O}(2-k,\mathcal{L})) \\ \to\; & \mathrm{Hom}(D_0, \mathcal{O}_{\epsilon,E}(k)) \to 0. \end{aligned} \tag{4.5.7}$$

Take $\Gamma_1^p(N)$-invariants and f-isotypic components. We have

$$\mathrm{Hom}_{\Gamma_1^p(N)}\left(D_0, P_{k-2} \otimes \epsilon^{-\frac{k-2}{2}}\right)^f = 0$$

by an argument as in Lemma 3.3.15.

Let

$$\delta_{\mathcal{L}} : \mathrm{Hom}_{\Gamma_1^p(N)}(D_0, \mathcal{O}(k)) \to H^1\left(\Gamma_1^p(N), \mathrm{Hom}\left(D_0, P_{k-2} \otimes \epsilon^{-\frac{k-2}{2}}\right)\right)$$

denote the coboundary map in the long exact sequence associated to (4.5.7). Recall that $\Phi_F^{\pm} \in \mathrm{Hom}_{\Gamma_1^p(N)}(D_0, \mathcal{O}(k))^f$. We will show that

$$\delta_{\mathcal{L}}(\Phi_F^{\pm}) = 0 \Leftrightarrow \mathcal{L} = -\mathcal{L}_O^{\pm}(f). \tag{4.5.8}$$

Thus if $\mathcal{L} \neq -\mathcal{L}_O^{\pm}(f)$, then $\Phi_F^{\pm} \notin \mathrm{Hom}_{\Gamma_1^p(N)}(D_0, \mathcal{O}(2-k,\mathcal{L}))$. Suppose on the other hand that $\mathcal{L} = -\mathcal{L}_O^{\pm}(f)$. We have an exact sequence of *finite dimensional* E-vector spaces:

$$\begin{aligned} 0 \;\to\; & \mathrm{Hom}_{\Gamma_1^p(N)}\left(D_0, P_{k-2} \otimes \epsilon^{-\frac{k-2}{2}}\right) \to \mathrm{Hom}_{\Gamma_1^p(N)}(D_0, \mathcal{O}(2-k,\mathcal{L})) \\ \to\; & \ker(\delta_{\mathcal{L}}) \to 0. \end{aligned} \tag{4.5.9}$$

The f-isotypic component of the leftmost non-trivial term in (4.5.9) is trivial; this implies that the rightmost nontrivial arrow induces an isomorphism on f-isotypic components. Since $\Phi_F^{\pm} \in (\ker \delta_{\mathcal{L}})^f$, we have that $\Phi_F^{\pm} \in \mathrm{Hom}_{\Gamma_1^p(N)}(D_0, \mathcal{O}(2-k,\mathcal{L}))^f$ as desired.

It remains to prove (4.5.8). Let $\tilde{\Phi}$ denote a lift of $\Phi_F^{\pm}$ via the surjection

$$\mathrm{Hom}(D_0, \mathcal{O}(2-k,\mathcal{L})) \to \mathrm{Hom}(D_0, \mathcal{O}(k)).$$

By definition, $\delta_{\mathcal{L}}(\Phi_F^{\pm})$ is the class of the 1-cocycle $\Delta_{\mathcal{L}}$ defined by

$$\gamma \mapsto \Delta_{\mathcal{L}}(\gamma) := \gamma(\tilde{\Phi}) - \tilde{\Phi} \in \mathrm{Hom}\left(D_0, P_{k-2}(E) \otimes \epsilon^{-\frac{k-2}{2}}\right).$$

(Recall that the map from $\mathcal{O}(2-k,\mathcal{L})$ to $\mathcal{O}(k)$ is the $(k-1)^{st}$ derivative.)

For $m \in D_0$ and $\gamma \in \Gamma_1^p(N)$, the functions $\gamma\tilde{\Phi}(m)$ and $\tilde{\Phi}(m)$ are $\log_{\mathcal{L}}$-rigid functions of a variable $T \in \mathcal{X}$, and the difference is a polynomial in T of degree at most $k-2$; around each point $z_0 \in \mathcal{X}$ we therefore have the representation

$$\begin{aligned} \gamma\tilde{\Phi}(\gamma^{-1}(m)) - \tilde{\Phi}(m) &= \sum_{i=0}^{k-2} \gamma\tilde{\Phi}(\gamma^{-1}m)^{(i)}(z_0)\frac{(T-z_0)^i}{i!} \\ &\quad - \sum_{i=0}^{k-2} \tilde{\Phi}(m)^{(i)}(z_0)\frac{(T-z_0)^i}{i!}, \end{aligned} \tag{4.5.10}$$

where the exponent (i) represents the ith derivative.

Now one checks that

(4.5.11)

$$\sum_{i=0}^{k-2} \gamma\tilde{\Phi}(\gamma^{-1}m)^{(i)}(z_0)\frac{(T-z_0)^i}{i!} = \gamma\left(\sum_{i=0}^{k-2} \tilde{\Phi}(\gamma^{-1}m)^{(i)}(\gamma^{-1}z_0)\frac{(T-\gamma^{-1}z_0)^i}{i!}\right).$$

Now we will correct $\Delta_{\mathcal{L}}$ by a coboundary in order to make it possible to finish the computation. Fix $Q \in \mathcal{X}$. Define $\psi \in \mathrm{Hom}(D_0, P_{k-2} \otimes \epsilon^{-\frac{k-2}{2}})$ by the formula

$$\psi(m) := \sum_{i=0}^{k-2} \tilde{\Phi}(m)^{(i)}(Q)\frac{(T-Q)^i}{i!},$$

and let $d\psi$ be the coboundary $d\psi(\gamma) = \gamma\psi - \psi$. Using (4.5.10) and (4.5.11) applied to $z_0 = \gamma Q$, one calculates

(4.5.12)

$$(d\psi(\gamma) - \Delta_{\mathcal{L}}(\gamma))(m) = \sum_{i=0}^{k-2} \tilde{\Phi}(m)^{(i)}(\gamma Q)\frac{(T-\gamma Q)^i}{i!} - \sum_{i=0}^{k-2} \tilde{\Phi}(m)^{(i)}(Q)\frac{(T-Q)^i}{i!}.$$

Now we apply $(\frac{d}{dz})$ to the function

$$\sum_{i=0}^{k-2} \tilde{\Phi}(m)^{(i)}(z)\frac{(T-z)^i}{i!}$$

and obtain

$$\tilde{\Phi}(m)^{(k-1)}(z)\frac{(T-z)^{k-2}}{(k-2)!} = \Phi_F^{\pm}(m)\frac{(T-z)^{k-2}}{(k-2)!}.$$

We can therefore express the right side of (4.5.12) as a Coleman line integral relative to $\mathcal{L}$:

$$d\psi(\gamma) - \Delta_{\mathcal{L}}(\gamma) = \frac{1}{(k-2)!}\int_Q^{\gamma Q} \Phi_F^{\pm}(m)(z)(T-z)^{k-2}dz.$$

To compare this with $c^{\pm}_{\mathcal{L},Q}$, we must first recall the fact that $P_{k-2}(E)$ and $\mathrm{Hom}(P_{k-2}(E), E))$ are isomorphic irreducible G-representations. Up to scalar multiplication, there is a unique isomorphism between these spaces. In fact, expanding the representative for $\Delta_{\mathcal{L}}(\gamma)$ we have computed above, we find

$$\Delta_{\mathcal{L}}(\gamma) \equiv \frac{1}{(k-2)!}\sum_{j=0}^{k-2}(-1)^j\binom{k-2}{j}T^j\int_Q^{\gamma Q} z^{k-2-j}\Phi_F^{\pm}(m).$$

In the isomorphism between $P_{k-2}(E)$ and $\mathrm{Hom}(P_{k-2}(E), E)$, This polynomial corresponds (up to multiples) to the cocycle

$$\delta_{\mathcal{L}}(\gamma)(x^j) = \int_Q^{\gamma Q} z^j\Phi_F^{\pm}(m)$$

which is nothing but $c^{\pm}_{\mathcal{L},Q}$. Consequently $\Delta_{\mathcal{L}}$ is cohomologous to zero precisely when $c^{\pm}_{\mathcal{L},Q} = 0$; the equivalence (4.5.8) follows from Proposition 4.5.4. $\square$

Note that, taking into account Theorem 4.5.2, we have in fact proven:

COROLLARY 4.5.13. *Let $\mathcal{L} \in E$. If $\mathcal{L} \neq -\mathcal{L}^{\pm}_{O}(f)$, then*

$$\mathrm{Hom}_{\Gamma_1^p(N)}(D_0, \mathcal{O}(2-k,\mathcal{L})^b)^{f,\pm} = 0.$$

If $\mathcal{L} = -\mathcal{L}^{\pm}_{O}(f)$, then we have isomorphisms

$$\mathrm{Hom}_{\Gamma_1^p(N)}(D_0, \mathcal{O}(2-k,\mathcal{L})^b)^{f,\pm} \cong \mathrm{Hom}_{\Gamma_1^p(N)}(D_0, \mathcal{O}(k)^b)^{f,\pm} = E\Phi^{\pm}_F.$$

4.6. Conclusion of proof of Breuil's Theorem.

LEMMA 4.6.1. *For any $\mathcal{L} \in E$, we have*

$$\mathrm{Hom}_{\tilde{\Gamma}_1^p(N)}(D_0, \mathcal{O}(2-k,\mathcal{L})^b \otimes \mathrm{nr}(w^{-1}))^f = \mathrm{Hom}_{\Gamma_0^p(N)}(D_0, \mathcal{O}(2-k,\mathcal{L})^b)^f.$$

PROOF. In view of Corollary 4.5.13, we must verify that $W_p\Phi^{\pm}_F = \lambda\Phi^{\pm}_F$ where W_p is a matrix in $\tilde{\Gamma}_1^p(N)$ which does not lie in $\Gamma_1^p(N)$. Such a matrix is given by $\begin{pmatrix} pu & v \\ Nps & t \end{pmatrix}$ where u, v, s, t are integers with $put - Nsv = 1$. The desired result then follows from

$$F(W_p^{-1}\alpha, z) = \frac{p^{k-1}}{a_p} F(\alpha, W_p z)(pNsz + pt)^{-k},$$

which itself follows from $f|_{W_p} = -a_p f$. For the details, see [**2**, Proposition 5.1.1] $\square$

Combining Theorem 4.3.1, Theorem 4.5.2, Lemma 4.6.1, and the duality between $B(2-k,\mathcal{L})$ and $\mathcal{O}(2-k,\mathcal{L})^b$, we find that

(4.6.2)

$$\mathrm{Hom}_{\mathrm{GL}_2(\mathbf{Q}_p)}(B(k,\mathcal{L}) \otimes \mathrm{nr}(w), \hat{H}^1_c(K_1^p(N)) \otimes E)^f \cong \mathrm{Hom}_{\Gamma_1^p(N)}(D_0, \mathcal{O}(2-k,\mathcal{L}))^f$$

for all $\mathcal{L} \in E$.

We may now deduce Breuil's Theorem:

THEOREM 4.6.3. *Let $\mathcal{L} \in E$. We have $\mathcal{L}^+_O = \mathcal{L}^-_O$ and if we let $\mathcal{L}_B$ denote this common value we have:*

$$H = \mathrm{Hom}_{\mathrm{GL}_2(\mathbf{Q}_p)}\left(B(k,\mathcal{L}) \otimes \mathrm{nr}(w), (\hat{H}^1_c(K_1^p(N)) \otimes E)^f\right)$$

satisfies

$$H \cong \begin{cases} 0 & \text{if } \mathcal{L} \neq -\mathcal{L}_B(f) \\ E\Phi^+_F + E\Phi^-_F & \text{if } \mathcal{L} = -\mathcal{L}_B(f). \end{cases}$$

PROOF. The equality of the $\mathcal{L}^{\pm}_O$ invariants is a consequence of the fact that the space H in the statement of the theorem carries an action by the Hecke algebra for f. Given one homomorphism h in this space, one can construct another by taking, for example, $T_\ell(h)$ for some ℓ prime to Np. These two homomorphisms are then independent (because, by the Eichler-Shimura relations, f picks out a two-dimensional subspace of the target). Thus H cannot be one-dimensional, and therefore the two $\mathcal{L}$ invariants must agree. The full result then follows from (4.6.2) and Corollary 4.5.13. $\square$

Breuil shows further that $E\Phi_F^+ + E\Phi_F^- \cong \sigma_p(f)^*$. He also proves that for all $\mathcal{L} \in E$, we have

$$\mathrm{Hom}_{\mathrm{GL}_2(\mathbf{Q}_p)}\left(B(k,\mathcal{L}) \otimes \mathrm{nr}(w), \hat{\pi}_p(f)\right) = 0$$

if $\mathcal{L} \neq -\mathcal{L}_B(f)$ and that if $k > 2$,

$$\mathrm{Hom}_{\mathrm{GL}_2(\mathbf{Q}_p)}\left(B(k,-\mathcal{L}_B(f)) \otimes \mathrm{nr}(w), \hat{\pi}_p(f)\right) = E.$$

In the introduction to this lecture, we asked Breuil's question: can we extract the $\mathcal{L}$-invariant of f from automorphic information? This result shows that this is, indeed, the case. This result is, essentially, a "formula" for $\mathcal{L}_B(f)$ that uses only representation theoretic information. Indeed, the space $\hat{\pi}_p(f)$ is born inside a large p-adic Banach space representation constructed from global automorphic data – the cohomology of modular curves. One might view it as the global p-adic automorphic representation attached to the modular form f. Breuil then shows that the $\mathcal{L}$ invariant of f identifies exactly which member of the family $B(k,-\mathcal{L})$ of Banach spaces occurs in this large representation. We should view $B(k,-\mathcal{L}_B(f))$ as the local representation at p associated to f in the big representation $\hat{\pi}_p(f)$.

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