E-functions and G-functions

Frits Beukers

February 26, 2008

Contents

1 Introduction

1.1 Transcendence and algebraic independence

Let K be a field and L an extension of K (not necessarily finite). An element $\alpha \in L$ is said to be *algebraic* over K if it is the zero of a non-trivial polynomial in $K[x]$. The monic polynomial $P(x) \in K[x]$ of minimal degree which has α as zero is called the *minimal polynomial* of α. The degree of α is defined to be the degree of $K[x]$.
An element $\alpha \in L$ is called *transcendental* over K if it is not algebraic over K.
A set of elements $\alpha_1, \ldots, \alpha_r \in L$ is called *algebraically independent* over K if $P(\alpha_1, \ldots, \alpha_r) \neq 0$ for any non-trivial polynomial $P(x_1 \ldots, x_r) \in K[x_1, \ldots, x_r]$.
Notice in particular that a transcendental element of L is called algebraically independent over K.
We say that the field L has *transcendence degree* r over K if the largest subset of L algebraically independent over K has size r. Notation $\mathrm{degtr}_K(L) = r$. When L contains no elements transcendental over K we set $\mathrm{degtr}_K(L) = 0$. When L contains arbitrarily large sets of algebraically independent elements we set $\mathrm{degtr}_K(L) = \infty$.
The most interesting problems, mostly the ones we are interested in at least, concern the case when $K = \mathbb{Q}$. It should be no surprise that proving transcendence of a real number (such as π and e) over $\mathbb{Q}$ is much harder than proving irrationality for example. It is therefore no surprise that in the beginning of the 19th century no examples of transcendental numbers were known. In 1844 Liouville gave the first examples of transcendental numbers.

Theorem 1.1.1 (Liouville 1844) *The number*

$$\alpha = \sum_{k=0}^{\infty} \frac{1}{2^{k!}}$$

is transcendental.

Proof Suppose α is algebraic. Let $P(x) \in \mathbb{Z}[x]$ be the minimal polynomial of α and assume its degree is D. It is a consequence of the mean value theorem there exists $M > 0$ such that for any $x \in [\alpha - 1, \alpha + 1]$ we have

$$|P(x)| = |P(x) - P(\alpha)| \leq M|x - \alpha|.$$

We can take $M = \max_{\xi \in [-1,1]} |P'(\alpha + \xi)|$. Let now α_n be the value of the truncated series

$$\alpha_n = \sum_{k=0}^{n} \frac{1}{2^{k!}}.$$

Notice that

$$\begin{aligned} |\alpha - \alpha_n| &= 2^{-(n+1)!} + 2^{-(n+2)!} + \cdots \\ &\leq 2^{-(n+1)!}(1 + 1/2 + 1/4 + \cdots) = 2 \cdot 2^{-(n+1)!} \end{aligned}$$

We derive two estimates, this time for $|P(\alpha_n)|$. Note that α_n is a rational number with denominator $2^{n!}$. Since P has at most finitely many zeros we have that $P(\alpha_n) \neq 0$ for sufficiently large n. Since $P(\alpha_n)$ is rational with denominator dividing $2^{Dn!}$ we conclude that $|P(\alpha_n)| \geq 2^{-Dn!}$ for sufficiently large n. On the other hand, by application of the mean value theorem and our series estimate,

$$|P(\alpha_n)| \leq M|\alpha_n - \alpha| \leq 2M2^{-(n+1)!}.$$

Putting the inequalities together we get that

$$2^{-Dn!} \leq 2M2^{-(n+1)!}$$

for sufficiently large n. Hence $2^{(n+1-D)n!} \leq 2M$ for sufficiently large n which gives a contradiction. $\square$

Of course this idea of proof can be applied to any number of the form

$$\sum_{n=0}^{\infty} q^{-a_n}$$

where $q \in \mathbb{Z}_{\geq 2}$ and a_n is an increasing sequence of positive integers such that $\lim_{n\to\infty} a_{n+1}/a_n = \infty$. This enabled Liouville to construct infinitely many examples of transcendental numbers.

Through the pioneering work of Cantor on set theory around 1874 it also became clear that 'almost all' real numbers are transcendental. This follows from the following two theorems.

Theorem 1.1.2 *The set of algebraic numbers is countable.*

Proof It suffices to show that the set $\mathbb{Z}[X]$ is countable. To any polynomial $P(X) = p_nX^n + p_{n-1}X^{n-1} + \cdots + p_1X + p_0 \in \mathbb{Z}[X]$ with $p_n \neq 0$ we assign the number $\mu(P) = n + |p_n| + |p_{n-1}| + \cdots + |p_0| \in \mathbb{N}$. Clearly for any $N \in \mathbb{N}$ the number of solutions to $\mu(P) = N$ is finite, because both the degree and the size of the coefficients are bounded by N. Hence $\mathbb{Z}[X]$ is countable. $\square$

Theorem 1.1.3 (Cantor) *The set of real numbers is uncountable.*

Proof We will show that the set of real numbers in the interval $[0, 1)$ is uncountable. Suppose that this set is countable. Choose an enumeration and denote the decimal expansion of the n-th real number by $0.a_{n1}a_{n2}a_{n3}\cdots$, where $a_{nm} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ for all n, m. Now consider the real number β whose decimal expansion reads $0.b_1b_2b_3\cdots$ where the b_i are chosen such that $b_i \neq a_{ii}$ for every i. This choice implies that β does not occur in our enumeration. Hence $[0, 1)$ is uncountable. $\square$

The principle of the proof of Theorem 1.1.3 is known as *Cantor's diagonal procedure* and it occurs in many places in mathematics.

Almost all real numbers being transcendental, it seems ironic that until the end of the 19-th century not a single 'naturally occurring' number was known to be transcendental. Only in 1873 Hermite showed that e is transcendental and in 1882 Lindemann proved π to be transcendental. Both these results are contained in the following Theorem.

Theorem 1.1.4 (Lindemann-Weierstrass, 1885) *Let $\alpha_1, \ldots, \alpha_n$ be distinct algebraic numbers contained in $\mathbb{C}$. Then the numbers $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are linearly independent over $\overline{\mathbb{Q}}$.*

In the 1920's C.L.Siegel extended this work widely to a much larger class of so-called E-functions which extend the exponential function. This development in transcendental number theory will form the focus of our attention in these lectures.

1.2 Developments not in this course

Of course we should point out that other developments took place as well in the same period. In his famous lecture of 1900 D.Hilbert asked whether numbers of the form a^b with a, b algebraic, $a \neq 0, 1$ and $b \notin \mathbb{Q}$, are transcendental. Specific examples are $2^{\sqrt{2}}$ and $i^{-2i} = e^{\pi}$. This problem, known as Hilbert's 7th problem, was considered to be very difficult by Hilbert, but already in the 1930's A.O.Gel'fond and Th.Schneider indepently developed techniques to solve this problem. So now we know,

Theorem 1.2.1 (Gel'fond, Schneider ,1934) *Let a, b be algebraic and suppose that $a \neq 0, 1$ and $b \notin \mathbb{Q}$. Then a^b is transcendental.*

Corollary 1.2.2 *Let α, β be two positive real algebraic numbers such that $\beta \neq 1$ and $\log\alpha/\log\beta \notin \mathbb{Q}$. Then $\log\alpha/\log\beta$ is transcendental.*

Proof Let $b = \log\alpha/\log\beta$ and suppose b is algebraic. Then, according to Theorem 1.2.1 the number $\alpha = \beta^b$ is transcental which is impossible since α is algebraic. $\square$

In 1966 A.Baker proved the following far-reaching generalisation of the Gel'fond-Schneider theorem.

Theorem 1.2.3 *Let $\alpha_1, \ldots, \alpha_n$ be non-zero algebraic numbers. If $\log\alpha_1, \ldots, \log\alpha_n$ are $\mathbb{Q}$-linear independent, then the numbers $1, \log\alpha_1, \ldots, \log\alpha_n$ are $\overline{\mathbb{Q}}$-linear independent.*

Baker also gave quantitative lower bounds for linear forms in logarithms of algebraic numbers which had profound consequences for the theory of diophantine equations. This work won him the Fields medal in 1974.
In the 15 years following Baker's work it turned out that the results could be vastly extended to linear forms in elliptic and abelian logarithms. Nowadays the Gel'fond-Schneider theory has grown into a field of its own in which large classes of numbers, ususally related to algebraic geometry, are known to be transcendental.
Despite all these developments the following conjecture is still unsolved.

Conjecture 1.2.4 (Schanuel) *Suppose that $b_1, \ldots, b_m$ are $\mathbb{Q}$-linearly independent. Then*

$$\mathrm{degtr}_{\mathbb{Q}}\mathbb{Q}(b_1, \ldots, b_m, e^{b_1}, \ldots, e^{b_m}) \geq m.$$

In the exercises we shall see that this conjecture entails the Lindemann-Weierstrass theorem and Baker's theorem and more. For example, e and π should be algebraically independent by Schanuel's conjceture (see the exercises) but it is not even known whether $e + \pi$ or $e\pi$ are irrational.

1.3 Values of Taylor series

A very natural question to ask is the following. Suppose we have a Taylor series expansion $f(z) = \sum_{n\ge 0} f_n z^n$ where the coefficients f_n are contained in an algebraic number field which we assume embedded in $\mathbb{C}$ and f has positive radius of convergence ρ, say.

Question 1.3.1 *Suppose $a \in \overline{\mathbb{Q}}$ with $0 < |a| < \rho$, is $f(a)$ transcendental?*

It turns out that in this generality the answer is 'no'. There exist Taylor series $f(z)$ which assume rational values at every algebraic point in its (positive) region of convergence, see []. In order to make any general statements at all we must make a number of restrictions on $f(z)$. In transcendence theory there are several such restrictions. The first one way studied by C.L.Siegel in 1929 []. After the success of the Lindemann-Weierstrass theorem Siegel wondered if the methods involved could be extended to functions like the Bessel function J_0, which is defined as

$$J_0(2iz) = \sum_{n=0}^{\infty} \frac{z^{2n}}{n!n!}.$$

Note that this series is reminiscent of the exponential function. Moreover, as is well known, Bessel functions satisfy a second order linear differential equation. Siegel called such functions E-functions and developed a transcendence theory for their values at algebraic points. In Section 3 we discuss them in more detail. Siegel also introduced a class of functions satisfying a linear differential equation and which is reminiscent of the gemetric series. He named them G-function. Transcendence theory for values of G-functions is much more rudimentary than that for E-functions. This is a bit unfortunate since values of G-functions occur quite naturally in arithmetic and geometry. Think of numbers like $\zeta(k)$ with k odd, or periods of differential forms on algebraic varieties defined over $\overline{\mathbb{Q}}$. For a more extensive discussion see Section 4.

1.4 Exercises

Exercise 1.4.1 *Show that* $\mathrm{degtr}_{\mathbb{Q}}(\overline{\mathbb{Q}}) = 0$.
Show, using elementary arguments a la Cantor, that $\mathrm{degtr}_{\mathbb{Q}}(\mathbb{C}) = \infty$

Exercise 1.4.2 *Show that a finite set of complex numbers is algebraically independent over $\mathbb{Q}$ if and only if they are independent over $\overline{\mathbb{Q}}$.*

Exercise 1.4.3 *For those who are familiar with the p-adic numbers $\mathbb{Q}_p$. Let $\mathbb{C}_p$ be the completion of the algebraic closure of $\mathbb{Q}_p$ Show that* $\mathrm{degtr}_{\mathbb{Q}_p}(\mathbb{C}_p) = \infty$.
The same question with Ω_p, the completion of the maximal unramified extension of $\mathbb{Q}_p$, instead of $\mathbb{C}_p$ seems to be even harder.

Exercise 1.4.4 *Show that Schanuel's conjecture implies the Lindemann-Weierstrass theorem.*

Exercise 1.4.5 *Let $a_1, \dots, a_n$ be algebraic numbers that are multiplicatively independent (i.e. no relation of the form $a_1^{k_1} \cdots a_n^{k_n} = 1$ with $k_i \in \mathbb{Z}$ not all zero). Show that Schanuel's conjecture implies that $\log a_1, \dots, \log a_n$ are algebraically independent.*

By way of contrast we remark that it is still not known whether $\log 2 \log 3$ is transcendental or not.

Exercise 1.4.6 *Show that Schanuel's conjecture implies the algebraic independence of e and π.*

Exercise 1.4.7 *Let us take $K = \mathbb{C}(z)$, the rational functions, as ground field.*

1. *Show that e^z is not in $\mathbb{C}(z)$.*
2. *Let $a_1, \dots, a_n$ be distinct complex numbers. Show that $e^{a_1 z}, \dots, e^{a_n z}$ are linearly independent over $\mathbb{C}(z)$.*
3. *Show that e^z is transcendental over $\mathbb{C}(z)$.*
4. *Let $a_1, \dots, a_n$ be complex numbers that are linearly independent over $\mathbb{Q}$. Then $e^{a_1 z}, \dots, e^{a_r z}$ are algebraically independent over $\mathbb{C}(z)$.*

2 Remarks on linear differential equations

2.1 Introduction

We let K be either the field of rational functions in one variable or $\mathbb{C}((z))$, the field of Laurent series in z, or the field of meromorphic functions in z.
An ordinary linear differential equation is an equation of the form

$$\partial^n y + p_1\partial^{n-1}y + \cdots + p_{n-1}\partial y + p_n y = 0, \quad p_1, \ldots, p_n \in K.$$

A *system of* n *first order equations* over K has the form

$$\partial \mathbf{y} = A\mathbf{y}$$

in the unknown column vector $\mathbf{y} = (y_1, \ldots, y_n)^t$ and where A is an $n \times n$-matrix with entries in K.
Note that if we replace $\mathbf{y}$ by $S\mathbf{y}$ in the system, where $S \in GL(n, K)$, we obtain a new system for the new $\mathbf{y}$,

$$\partial \mathbf{y} = (S^{-1}AS + S^{-1}\partial S)\mathbf{y}.$$

Two $n \times n$-systems with coefficient matrices A, B are called *equivalent over* K if there exists $S \in GL(n, K)$ such that $B = S^{-1}AS + S^{-1}\partial S$.
It is well known that a differential system can be rewritten as a system by putting $y_1 = y, y_2 = y', \ldots, y_n = y^{(n-1)}$. We then note that $y_1' = y_2, y_2' = y_3, \ldots, y_{n-1}' = y_n$ and finally, $y_n' = -p_1 y_n - p_2 y_{n-1} - \ldots - p_n y_1$. This can be rewritten as

$$\frac{d}{dz}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -p_n & -p_{n-1} & -p_{n-2} & \cdots & -p_1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

There is also a converse statement.

Theorem 2.1.1 (Cyclic vector Lemma) *Any system of linear first order differential equations over* K *is equivalent over* K *to a system which comes from a differential equation.*

2.2 Local theory

In this section our differential field will be $\mathbb{C}((z))$. We shall denote the derivation $\frac{d}{dz}$ by ∂ and the derivation $z\frac{d}{dz}$ by θ.
Consider the linear differential equation of order n,

$$\partial^n y + p_1(z)\partial^{(n-1)}y + \cdots + p_{n-1}(z)\partial y + p_n(z)y = 0, \tag{1}$$

with $p_i \in \mathbb{C}((z))$. If $z = 0$ is not a pole of any p_i it is called a *regular point* of (1), otherwise it is called a *singular point* of (1). The point $z = 0$ is called a *regular singularity* if p_i has a pole of order at most i for $i = 1, \ldots, n$.

Another way of characterising a regular singularity is by rewriting (1) with respect to the derivation θ. Multiply (1) with z^n and use $z^r D^r = \theta(\theta-1)(\theta-r+1)$ to obtain an equation of the form

$$\theta^n y + q_1(z)\theta^{n-1}y + \cdots + q_{n-1}(z)\theta y + q_n(z)y = 0. \tag{2}$$

The condition for $z=0$ to be a regular singularity comes down to $q_i \in \mathbb{C}[[z]]$ for all i.

Theorem 2.2.1 (Cauchy) *Suppose* 0 *is a regular point of* (1). *Then there exist* n $\mathbb{C}$-*linear independent Taylor series solutions* $f_1, \ldots, f_n \in \mathbb{C}[[z]]$. *Any Taylor series solution of* (1) *is a* $\mathbb{C}$-*linear combination of* $f_1, \ldots, f_n$. *Moreover, if the coefficients of* (1) *have positive radius of convergence, the same holds for* $f_1, \ldots, f_n$.

When the point is a regular singular point one can still make a statement, although there is in general not a basis of Taylor series solutions any more. Instead we have to look for local solutions of the form $z^\rho y(z)$ where $y(z)$ is a Taylor series in z.
In the following theorem we shall consider expressions of the form z^A where A is a constant $n\times n$ matrix. This is short hand for

$$z^A = \exp(A\log z) = \sum_{k\ge 0}\frac{1}{k!}A^k(\log z)^k.$$

In particular z^A is an $n \times n$ matrix of multivalued functions around $z=0$. Examples are,

$$z^{\begin{pmatrix}1/2 & 0\\ 0 & -1/2\end{pmatrix}} = \begin{pmatrix} z^{1/2} & 0\\ 0 & z^{-1/2}\end{pmatrix}, \qquad z^{\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}} = \begin{pmatrix} 1 & \log z\\ 0 & 1\end{pmatrix}.$$

In the following Theore we also need the so-called *indicial equation* of a linear differential equation with a regular singularity. It reads

$$X^n + q_1(0)X^{n-1} + q_2(0)X^{n-2} + \cdots + q_{n-1}(0)X + q_n(0) = 0$$

where the $q_i(z)$ are the coefficients from (2).

Theorem 2.2.2 (Fuchs) *Suppose that* $z=0$ *is a regular singularity of the equation (1). Then there exist Taylor series* $u_1, \ldots, u_n$ *and an upper triangular matrix* B *with constant entries in Jordan normal form, such that a basis of solutions of (1) is given by*

$$(u_1, u_2, \ldots, u_n)z^B.$$

Any eigenvalue ρ *of* B *is the minimum of all roots of the indicial equation of the form* $\rho, \rho+1, \rho+2, \ldots$.
Moreover, if the coefficients of (1) have positive radius of convergence, the same holds for the entries $u_i(z)$.

The roots of the indicial equation are called the *local exponents* at $z=0$ of the system.

Exercise 2.2.3 *Show that the local exponents at a regular point read* $0, 1, \ldots, n-1$.

Exercise 2.2.4 *Suppose that the local exponents at $z = 0$ are $0, 1, \ldots, n-1$ and that to each exponents there corresponds a holomorphic solution. Then $z = 0$ is a regular point (i.e. non-singular).*

Exercise 2.2.5 *Consider the linear differential equation*

$$(z^3 + 11z^2 - z)y'' + (3z^2 + 22z - 1)y' + (z + 3)y = 0.$$

Show that the local exponents at $z = 0$ are $0, 0$ and determine the recursion relation for the holomorphic solution near $z = 0$. Determine also the first few terms of the expansions of a basis of solutions near $z = 0$.

2.3 Fuchsian equations

In this section our differential field will be $\mathbb{C}(z)$, the field of rational functions in z and we shall consider our differential equations and $n \times n$-systems over this field.
Consider the linear differential equation

$$y^{(n)} + p_1(z)y^{(n-1)} + \cdots + p_{n-1}(z)y' + p_n(z)y = 0, \qquad p_i(z) \in \mathbb{C}(z) \tag{3}$$

To study this differential equation near any point $P \in \mathbb{P}^1$ we choose a local parameter $t \in \mathbb{C}(z)$ at this point (usually $t = z - P$ if $P \in \mathbb{C}$ and $t = 1/z$ if $P = \infty$), and rewrite the equation with respect to the new variable t. We call the point P a regular point or a regular singularity if this is so for the equation in t at $t = 0$. It is not difficult to verify that a point $P \in \mathbb{C}$ is regular if and only if the p_i have no pole at P. It is a regular singularity if and only if $\lim_{z \to P}(z - P)^i p_i(z)$ exists for $i = 1, \ldots, n$. The point ∞ is regular or a regular singularity if and only if $\lim_{z\to\infty} z^i p_i(z)$ exists for $i = 1, \ldots, n$.

Definition 2.3.1 *A differential equation over $\mathbb{C}(z)$ or a system of first order equations over $\mathbb{C}(z)$ is called* <u>*Fuchsian*</u> *if all points on $\mathbb{P}^1$ are regular or a regular singularity.*

Let $P \in \mathbb{P}^1$ be any point which is regular or a regular singularity. Let t be a local parameter around this point and rewrite the equation (3) with respect to the variable t. The corresponding indicial equation will be called the indicial equation of (3) at P. The roots of the indicial equation at P are called the *local exponents* of (3) at P.
This procedure can be cumbersome and as a shortcut we use the following lemma to compute indicial equations.

Lemma 2.3.2 *Let $P \in \mathbb{C}$ be a regular point or regular singularity of* (3)*. Let $a_i = \lim_{z\to P}(z - P)^i p_i(z)$ for $i = 1, \ldots, n$. The indicial equation at P is given by*

$$X(X - 1)\cdots(X - n + 1) + a_1X(X - 1)\cdots(X - n + 2) + \cdots + a_{n-1}X + a_n = 0.$$

When ∞ is regular or a regular singularity, let $a_i = \lim_{z\to\infty} z^i p_i(z)$ for $i = 1, \ldots, n$. The indicial equation at ∞ is given by

$$\begin{aligned}&X(X + 1)\cdots(X + n - 1) - a_1X(X + 1)\cdots(X + n - 2) + \cdots\\ &+(-1)^{n-1}a_{n-1}X + (-1)^n a_n = 0.\end{aligned}$$

Proof. Exercise □

3 E-functions

3.1 Definition

In generalising the Lindemann-Weierstrass theorem, C.L.Siegel in [17] introduced entire analytic functions whose power series expansion is very reminiscent of the exponential series and which, moreover, satisfy a linear differential equation.
In the definition below we assume that the algebraic numbers are embedded in $\mathbb{C}$. For any element $\alpha \in \overline{\mathbb{Q}}$ we define $\overline{|\alpha|}$ to be maximum of the absolute values of all conjugates of α. We call it the *size* of α. By $\mathrm{den}(\alpha)$ we denote the *denominator* of α, the smallest positive integer d such that $d\alpha$ is an algebraic integer. For any set of $\alpha_1, \ldots, \alpha_r$ we denote by $\mathrm{den}(\alpha_1, \ldots, \alpha_r)$ the lowest common multiple of the denominators of $\alpha_1, \ldots, \alpha_r$.
An entire function $f(z)$ given by a powerseries

$$\sum_{n=0}^{\infty} \frac{a_k}{k!} z^k$$

with $a_i \in \overline{\mathbb{Q}}$ for all i, is called an E-function if

1. $f(z)$ satisfies a linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$.
2. Both $\overline{|a_n|}$ and the common denominator $\mathrm{den}(a_0, \ldots, a_n)$ are bounded by an exponential bound of the form C^n, where $C > 0$ depends only on f.

Examples:

$$\begin{aligned} \exp(z) &= \sum_{k=0}^{\infty} \frac{z^k}{k!} \\ J_0(-z^2) &= \sum_{k=0}^{\infty} \frac{z^{2k}}{k!k!} \\ f(z) &= \sum_{k=0}^{\infty} \frac{a_k}{k!} z^k \end{aligned}$$

where $a_0 = 1, a_1 = 3, a_2 = 19, a_3 = 147, \ldots$ are the Apéry numbers corresponding to Apéry's irrationality proof of $\zeta(2)$. They are determined by the recurrence relation $(n+1)^2 a_{n+1} = (11n^2 - 11n + 3)a_n - n^2 a_{n-1}$.
The corresponding differential equations read

$$\begin{aligned} y' - y &= 0 \\ zy'' + y' - 4zy &= 0 \\ z^2y''' - (11z^2 - 3z)y'' - (z^2 + 22z - 1)y' - (z+3)y &= 0 \end{aligned}$$

Although in the definition the coefficients a_n are in $\overline{\mathbb{Q}}$, there exists a number field K such that $a_n \in K$ for all n. The reason is that the differential equation satisfied by f has a finite number of coefficients in $\overline{\mathbb{Q}}(z)$. The field generated by the coefficients of these rational functions is a finite extension of $\mathbb{Q}$.

In the original definition Siegel used instead of the exponential bound C^N in condition 2) the (seemingly) less restrictive $c_\epsilon(N!)^\epsilon$ for every $\epsilon > 0$. However, it is conjectured that the bound $(N!)^\epsilon$ together with the fact that the function satisfies a linear differential equation is enough to garantee that we have the exponential bound C^N. Not a single example is known to the contrary. Therefore we shall stick to our definition above.

3.2 Ring structure

Before going to their values we mention a few properties.

Proposition 3.2.1 *The E-functions form a ring.*

Proof. We need to show that both the sum and product of two E-functions f, g is again an E-function. The fact that f satisfies a linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$ is equivalent to the fact that the $\overline{\mathbb{Q}}(z)$-linear vector space generated by $f, f', f'', \ldots$ is finite dimensional. The same holds for g and its derivatives. As a consequence the function $f + g$ and its derivates span a finite dimensional vector space, hence $f + g$ satisfies a linear differential equation. Similarly it is easy to verify that fg and its derivatives span a finite dimensional space.
So condition (1) for $f + g$ and fg is verified. It remains to verify condition (2).
To make everything simple we restrict ourselves to the case when the coefficients of f, g are in $\mathbb{Q}$. Suppose

$$f = \sum_{n\ge0} \frac{f_n}{n!} z^n, \quad g = \sum_{n\ge0} \frac{g_n}{n!} z^n$$

where $f_n, g_n \in \mathbb{Q}$ for all n. Clearly $f+g$ and fg have their coefficients again in $\overline{\mathbb{Q}}$. Let us choose $C > 0$ such that $\overline{|f_n|} < C^n$ and $\overline{|g_n|} < C^n$, $\mathrm{den}(f_0, \ldots, f_n) < C^n$, $\mathrm{den}(g_0, \ldots, g_n) < C^n$ for all n.
The coefficients of $f + g$ read $(f_n + g_n)/n!$. Clearly $\overline{|f_n + g_n|} < 2C^n$ and $\mathrm{den}(f_0 + g_0, \ldots, f_n + g_n) < C^{2n}$. Hence condition (2) is satisfied for $f + g$.
The n-th coefficient of fg (ignoring $n!$) reads

$$h_n = \sum_{k=0}^{n} \binom{n}{k} f_k g_{n-k}.$$

Notice that

$$\overline{|h_n|} \le 2^n \max_{k,n-k} \overline{|f_k|} \cdot \overline{|g_{n-k}|} < 2^n C^n$$

and

$$\mathrm{den}(h_0, \ldots, h_n) \le \mathrm{den}(g_0, g_1, \ldots, g_n)\mathrm{den}(f_0, f_1, \ldots, f_n) \le C^{2n}.$$

Again condition (2) is satisfied.

$\square$

3.3 Transcendence results

Siegel, around 1929, developed methods to prove transcendence and algebraic independence of values of E-functions at algebraic arguments. An important technical condition in Siegel's work was the so-called normality condition, which presented significant obstacles. Later this technical condition was circumvented by A.B.Shidlovsky in [20].
The theorems on values of E-functions are usually formulated in terms of the first order systems that correspond to a linear differential equation. In this section the standard notation for an $n \times n$-system of first order equations will be

$$\mathbf{y}'(z) = A(z)\mathbf{y}(z) \tag{A}$$

where $A(z)$ is an $n \times n$-matrix with coefficients in $\mathbb{C}(z)$ and whose common denominator we denote by $T(z)$.
We now have the following Theorem.

Theorem 3.3.1 (Siegel-Shidlovskii, 1929, 1956) . *Let $(f_1(z), \ldots, f_n(z)$ be a solution vector of a system of first order equations of the form (A) and suppose that the $f_i(z)$ are E-functions. Let $T(z)$ be the common denominator of the entries of $A(z)$. Let $\alpha \in \overline{\mathbb{Q}}$ and $\alpha T(\alpha) \neq 0$. Then*

$$\text{degtr}\overline{\mathbb{Q}}(f_1(\alpha), f_2(\alpha), \ldots, f_n(\alpha)) = \text{degtr}\mathbb{C}(z)(f_1(z), f_2(z), \ldots, f_n(z))$$

In particular, if the $f_i(z)$ are algebraically independent over $\mathbb{C}(z)$ then the values at $z = \alpha$ are algebraically independent over $\overline{\mathbb{Q}}$ (or $\mathbb{Q}$, which amounts to the same). After this theorem became available, much effort was spent into showing that large classes of E-functions are algebraically independent over $\mathbb{C}(z)$. See the book by Shidlovskii [19] and the paper [7], where differential galois theory was applied for the first time in these questions.
Although the Siegel-Shidlovskii theorem is a very strong one, it does not answer a question such as the following. Suppose that the $f_i(z)$ are linearly independent over $\mathbb{C}(z)$, are the values at $z = \alpha$ linearly independent over $\overline{\mathbb{Q}}$?
Quite recently in [8] this question was answered affirmatively.

Theorem 3.3.2 (Beukers, 2004) *Let notations be as in Theorem 3.3.1. Suppose that the $f_i(z)$ are $\mathbb{C}(z)$-linear independent. Then for any $\alpha \in \overline{\mathbb{Q}}$ which is not a zero of $zT(z)$, the numbers $f_i(\alpha)$ are $\overline{\mathbb{Q}}$-linear independent.*

Let us write $\mathbf{f}(z) = (f_1(z), \ldots, f_n(z))^t$ and $A(z) = M(z)/T(z)$. Suppose that α is a zero of $T(z)$. Then clearly it follows from the system of equations that, since $T(\alpha)\mathbf{f}'(\alpha) = \mathbf{0}$, we get $\mathbf{0} = M(\alpha)\mathbf{f}(\alpha)$. Since $M(\alpha) \neq 0$ this gives non-trivial $\overline{\mathbb{Q}}$-linear relations between the numbers $f_i(\alpha)$.
Notice that Theorem 3.3.2 is a natural generalisation of the Lindemann Weierstrass Theorem. When $\beta_1, \ldots, \beta_n$ are distinct algebraic numbers, the functions $e^{\beta_1 z}, \ldots, e^{\beta_n z}$ are $\mathbb{C}(z)$-linear independent and they satisfy a system of first order linear differential equations with constant coefficients. Hence their values at $z = 1$, that is $e^{\beta_1}, \ldots, e^{\beta_n}$, are $\overline{\mathbb{Q}}$-linear independent.
In fact, Theorem 3.3.2 is a special case of a much more general Theorem.

Theorem 3.3.3 (Beukers, 2004) *Let $f_1(z), \dots, f_n(z)$ be E-functions which satisfy a system of n first order equations. Then for any $\xi \in \overline{\mathbb{Q}}$ which is not 0 or a singularity of the system of differential equations, the following statement holds. To any relation of the form $P(f_1(\xi), \dots, f_n(\xi)) = 0$ where $P \in \overline{\mathbb{Q}}[X_1, \dots, X_n]$ there exists a $Q \in \overline{\mathbb{Q}}[z, X_1, \dots, X_n]$ such that $Q(z, f_1(z), \dots, f_n(z)) \equiv 0$ and*

$$P(X_1, \dots, X_n) = Q(\xi, X_1, \dots, X_n)$$

Roughly speaking, any algebraic relation over $\overline{\mathbb{Q}}$ between $f_1(\xi), \dots, f_n(\xi)$ at some non-zero, non-singular point $\xi \in \overline{\mathbb{Q}}$ comes from specialisation at $z = \xi$ of some functional algebraic relation between $f_1(z), \dots, f_n(z)$ over $\overline{\mathbb{Q}}(z)$.
In [15] a weaker version of this Theorem was proven using the Siegel-Shidlovskii Theorem. The weaker part consisted of the fact that there is an extra condition $\xi \notin S$, where S is a certain finite set of points.

3.4 Y.André's theorem

In many examples of E-functions the corresponding differential equation has singularities only at $z = 0$ (and $z = \infty$ by default). One may wonder if this is always the case. In a brilliant paper Y.André [3] confirmed this up to a certain level.

Theorem 3.4.1 (Y.André, 2000) *Let $f(z)$ be an E-function. Then $f(z)$ satisfies a differential equation of the form*

$$z^m y^{(m)} + \sum_{k=0}^{m-1} z^k q_k(z) y^{(k)} = 0$$

where $q_k(z) \in \overline{\mathbb{Q}}[z]$ has degree $\le m - k$.

So any E-function satisfies a linear differential equation having only $z = 0, \infty$ as singularities. We note that this differential equation need not be the lowest order differential equation which admits $f(z)$ as a solution. We call the latter the *minimal differential equation* satisfied by $f(z)$. This differential equation may have singularities, which are then necessarily apparent singularities.
For example, the function $(z-1)e^z$ is an E-function, and its minimal differential equation reads $(z-1)f' = zf$. So we have a singularity at $z = 1$. The equation refered to in André's theorem might be $f'' - 2f' + f = 0$.
The startling feature of André's theorem is that it allows us to give transcendence proofs. This is explained by the following two corollaries.

Corollary 3.4.2 *Let $f(z)$ be an E-function with coefficients in $\mathbb{Q}$ and suppose that $f(1) = 0$. Then 1 is an apparent singularity of the minimal differential equation satisfied by f.*

As we just noted, the simplest example is of course $f = (z-1)e^z$, an E-function which vanishes at $z = 1$. Its minimal differential equation is $(z-1)f' = zf$.

Proof. Consider $f(z)/(1-z)$. This is an entire analytic function. Suppose that

$$f(z)=\sum_{n\ge 0}\frac{f_n}{n!}z^n.$$

Then the power series of $f(z)/(1-z)$ reads

$$\frac{f(z)}{1-z}=\sum_{n\ge 0}\frac{g_n}{n!}z^n$$

where

$$g_n=\sum_{k=0}^{n}\frac{n!}{k!}f_k.$$

Suppose that the common denominator of $f_0,\ldots,f_n$ and the sizes $|f_n|$ are bounded by C^n for some $C>0$. Then clearly the common denominator of $g_0,\ldots,g_n$ are again bounded by C^n. To estimate the size of $|g_n|$ we use the fact that $0=f(1)=\sum_{k\ge 0}f_k/k!$. More precisely,

$$\begin{aligned}|g_n| &= \left|-\sum_{k>n}n!f_k/k!\right|\\ &\le \sum_{k>n}|f_k|/(n-k)!\\ &\le \sum_{k>n}C^k/(n-k)! < C^ne^C\end{aligned}$$

So $|g_n|$ is exponentially bounded in n. Hence $f(z)/(1-z)$ is an E-function. Notice that this argument only works if $f(z)$ is an E-function with coefficients in $\mathbb{Q}$.
By Andre's theorem $f(z)/(1-z)$ satisfies a differential equation without singularity at $z=1$. Hence its minimal differential equation has a basis of analytic solutions at $z=1$. This means that the original differential equation for $f(z)$ has a basis of analytic solutions all vanishing at $z=1$. So $z=1$ is apparent singularity. □

Corollary 3.4.3 *π is transcendental.*

Proof. Suppose $\alpha:=2\pi i$ algebraic. Then the E-function $e^{\alpha z}-1$ vanishes at $z=1$. The product over all conjugate E-functions is an E-function with rational coefficients vanishing at $z=1$. So the above corollary applies. However linear forms in exponential functions satisfy differential equations with constant coefficients, contradicting existence of a singularity at $z=1$. □

In [3] it is shown that ideas like the one above lead to a completely different proof of the Siegel-Shidlovskii theorem. However, by a combination of André's Theorem and differential galois theory one can show more.

Theorem 3.4.4 (Beukers, 2004) *Let $f(z)$ be an E-function and suppose that $f(\xi)=0$ for some $\xi\in\overline{\mathbb{Q}}^*$. Then ξ is an apparent singularity of the minimal differential equation satisfied by f.*

In [8] it is shown that this Theorem implies Theorem 3.3.3.

3.5 Basic E-functions

We have seen above that $(z-1)e^z$ is an E-function which vanishes at $z=1$ and whose minimal differential equation has a singularity at $z=1$. All this seems a bit artificial because it is clear that we should be looking at the function e^z. A similar phenomenon occurs in general.

Theorem 3.5.1 *Let $\mathbf{f}(z) = (f_1(z), \ldots, f_n(z))$ be E-function solution of system of n first order equations. Then there exists $n \times n$- matrix B with entries in $\overline{\mathbb{Q}}[z]$ and $\det(B) \neq 0$ and E-functions $\mathbf{e}(z) = (e_1(z), \ldots, e_n(z)$ such that $\mathbf{f}(z) = B\mathbf{e}(z)$ and $\mathbf{e}(z)$ satisfies system of equations with singularities in the set $\{0, \infty\}$.*

3.6 Exercises

Exercise 3.6.1 *Let $k_1, \ldots, k_p, m_1, \ldots, m_q$ be positive integers and suppose that $d = m_1 + \cdots + m_q - (k_1 \cdots + k_p)$ is positive. Define*

$$f(z) = \sum_{n \geq 0} \frac{(k_1 n)!(k_2 n)! \cdots (k_p n)!}{(m_1 n)!(m_2 n)! \cdots (m_q n)!} z^{kn}.$$

1. *Show that f satisfies a linear differential equation of order $m_1 + \cdots + m_q$.*
2. *Show that f is an E-function*

3.7 An example of relations between E-functions

Example

$$f(z) = \sum_{k=0}^{\infty} \frac{((2k)!)^2}{(k!)^2(6k)!} z^k$$

and $f(z^4)$ is an E-function satisfying a differential equation of order 5. The differential galois group is $SO(5, \mathbb{C})$. Dimension of its orbits is 4 and we have a quadratic form Q with coefficients in $\mathbb{Q}(z)$ such that

$$Q(f, f', f'', f''', f'''') = 1$$

Explicitly,

$$f(z) = \sum_{k=0}^{\infty} \frac{((2k)!)^2}{(k!)^2(6k)!} (2916z)^k$$

satisfies

$$\mathbf{F}^t \mathcal{Q} \mathbf{F} = (z)$$

where

$$\mathbf{F} = \begin{pmatrix} f(z) \\ Df(z) \\ D^2 f(z) \\ D^3 f(z) \\ D^4 f(z) \end{pmatrix}, \qquad D = z\frac{d}{dz}$$

and

$$\mathcal{Q} = \begin{pmatrix} z - 324z^2 & -18z & 198z & -486z & 324z \\ -18z & -\frac{10}{9} & \frac{23}{2} & -28 & 18 \\ 198z & \frac{23}{2} & -120 & 297 & -198 \\ -486z & -28 & 297 & -729 & 486 \\ 324z & 18 & -198 & 486 & -324 \end{pmatrix}$$

4 G-functions

4.1 Definition

After the successful work on transcendence of values of E-functions the question arose whether Siegel's method could also be applied to powerseries that are reminiscent of geometric series (much like the resemblance between E-functions and e^z).
An analytic function $f(z)$ given by a powerseries

$$\sum_{n=0}^{\infty} a_k z^k$$

with $a_i \in \overline{\mathbb{Q}}$ for all i and positive radius of convergence, is called a G-function if

1. $f(z)$ satisfies a linear differential equation with coefficients in $\mathbb{Q}(z)$.

2. Both $\overline{|a_n|}$ and the common denominators $\mathrm{den}(a_0, \ldots, a_n)$ are bounded by an exponential bound of the form C^n, where $C > 0$ depends only on f.

Examples:

Theorem 4.1.1 *The following Taylor series $f(z) = \sum_{n\geq 0} f_n z^n$ with $f_n \in \mathbb{Q}$ are G-functions.*

1. *$f(z)$ is algebraic over $\mathbb{Q}(z)$ (Eisenstein theorem).*

2. *$f(z) = {}_2F_1(\alpha, \beta, \gamma|z)$, a Gauss hypergeometric series with rational parameters α, β, γ.*

3. *$f(z) = L_k(z) = \sum_{n\geq 1} \frac{z^n}{n^k}$, the k-th polylogarithm.*

4. *$f(z) = \sum_{k=0}^{\infty} a_k z^k$ where $a_0 = 1, a_1 = 3, a_2 = 19, a_3 = 147, \ldots$ are the Apéry numbers corresponding to Apéry's irrationality proof of $\zeta(2)$. They are determined by the recurrence relation $(n+1)^2 a_{n+1} = (11n^2 - 11n + 3)a_n - n^2 a_{n-1}$.*

4.2 Periods

Values of G-functions at algebraic points play an important role in arithmetic and algebraic geometry. To illustrate this point we remind you of the values of the Riemann zeta-function

$$\zeta(k) = \frac{1}{1^k} + \frac{1}{2^k} + \frac{1}{3^k} + \cdots$$

These are the values at $z = 1$ of the polylogarithm $L_k(z)$ defined above. For even k it is known that $\zeta(k)/\pi^k$ is rational (Euler), but for odd k almost nothing is known. The irrationality of $\zeta(3)$ was fairly recently proved by R.Apéry in 1978 (see [6] and [16]) and in 2000 T.Rivoal, [5], showed that there are infinitely many irrational numbers among $\zeta(k)$ with k odd.

In a slightly more general setting values of Dirichlet L-series at integer points can also be included as values of G-functions. For example, it is not known if the so-called Catalan constant

$$L(2, \chi_4) = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots$$

is irrational or not. Here χ_4 denotes the odd Dirichlet character with conductor 4.
Lastly, periods of differential forms on algebraic varieties defined over $\overline{\mathbb{Q}}$ are also a source of values of G-functions. To illustrate this point we recall the Euler integral for the hypergeometric function

$${}_2F_1(1/5, 4/5, 8/5|z) = \frac{\Gamma(8/5)}{\Gamma(4/5)^2} \int_0^1 \frac{dx}{x^{1/5}(1-x)^{1/5}(1-zx)^{1/5}}.$$

This integral can be interpreted as a period (integral over a closed loop) of the differential form dx/y on the algebraic curve $y^5 = x(1-x)(1-zx)$.
Transcendence theory for periods of 1-forms on algebraic curves or abelian varieties is quite well developed as a result of farreaching generalisations of Baker's method for linear forms in logarithms. Unfortunately, irrationality and transcendence for periods of n-forms with $n \geq 2$ is mostly terra incognita at this moment. Siegel's theory for G-functions gives us very limited information at the moment.
Zagier and Kontsevich [14] have formulated a number of conjectures on transcendence and algebraic (in)dependence of periods.

4.3 Transcendence results

As we indicated previously, irrationality and transcendence results for values of G-functions are not nearly as nice as for E-functions. In fact, this cannot be expected since we have to make exceptions a priori. For example, algebraic functions (over $\mathbb{Q}(z)$) are G-functions but, trivially, their values at algebraic arguments are again algebraic. Furthermore, Gaussian hypergeometric functions sometimes assume algebraic values at algebraic arguments. Here are two well-known examples

$${}_2F_1\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2} \middle| \frac{1323}{1331}\right) = \frac{3}{4}\sqrt[4]{11}.$$

$${}_2F_1\left(\frac{1}{12}, \frac{7}{12}, \frac{2}{3} \middle| \frac{64000}{64009}\right) = \frac{2}{3}\sqrt[6]{253}.$$

Transcendence of values of hypergeometric functions at algebraic arguments was described in [21] and is based on Baker-Wüstholz theory. Unfortunately there is a gap in the algebraic-geometric part of the proof of the main theorem which was only later filled in by Yafaev (see the paper by Edixhoven and Yafaev, [12]).
Roughly speaking the general approach in Siegel's sense doesn't give transcendence results, but only linear independence of G-function values at arguments sufficiently close to the origin.

4.4 Irrationality results

On pages 239-241 in [17] Siegel indicates what sort of result might be attainable using his approach. In spite of that the first elaborated results became only available in the 1970's with the work of Galochkin, [13] and slightly later, E. Bombieri, [9]. In these papers a very strong condition was required in order for Siegel's method to work. They are known as Galochkin's condition (from [13]) and Bombieri's Fuchsian of arithmetic type (from [9]). It turns out that these conditions are equivalent (see [11], Ch VII, Thm 2.1) and we formulate Galochkin's condition now.
Just as with E-functions we start with an $n \times n$-system of linear differential equations

$$\frac{d\mathbf{y}}{dz} = G\mathbf{y}$$

where G is an $n\times n$-matrix with entries in $\overline{\mathbb{Q}}(z)$. By induction we define for $s = 1, 2, 3, \ldots$ the iterated $n \times n$-matrices G_s by

$$\frac{d^s\mathbf{y}}{dz^s} = G_s\mathbf{y}.$$

Notice that

$$G_{s+1} = G_sG + \frac{dG_s}{dz}.$$

Let $T(z)$ be the common denominator of all entries of G. Then, for every s, the entries of $T(z)^sG_s$ are polynomials. Denote the least common denominator of all coefficients of all entries of $T(z)^mG_m/m!$ $(m = 1, \ldots, s)$ by q_s.

Definition 4.4.1 *With notation as above, we say that the system* $\mathbf{y}'(z) = G(z)\mathbf{y}(z)$ *satisfies Galochkin's condition if and only if there exists* $C > 0$ *such that* $q_s < C^s$ *for all* $s \geq 1$.

A typical result obtainable with G-functions via Siegel's method is the following.

Theorem 4.4.2 (Galochkin) *Let* $(f_1(z), \ldots, f_n(z)$ *be a solution vector of a system of first order equations of the form* $\mathbf{y}' = G\mathbf{y}$ *and suppose that the* $f_i(z)$ *are G-functions with coefficients in* $\mathbb{Q}$. *Suppose also that* $f_1(z), \ldots, f_n(z)$ *are linearly independent over* $\mathbb{Q}(z)$ *and that the system satisfies Galochkin's condition. Then there exists* $C > 0$ *such that* $f_1, (a/b), \ldots, f_n(a/b)$ *are* $\mathbb{Q}$*-linear independent whenever* $a, b \in \mathbb{Z}$ *and* $b > C|a|^{n+1} > 0$.

In [9] Bombieri has extended results of this type in the direction of algebraic independence of bounded degree and with algebraic coefficients from a number field.
A major innovation was made by the Chudnovsky's in 1984, [10], who showed that Galochkin's condition was in many cases automatically satisfied.

Theorem 4.4.3 (Choodnovsky) *Let* $(f_1(z), \ldots, f_n(z)$ *be a solution vector consisting of G-functions of a system of first order equations of the form* $\mathbf{y}' = G\mathbf{y}$. *Suppose that* $f_1(z), \ldots, f_n(z)$ *are linearly independent over* $\overline{\mathbb{Q}}(z)$. *Then the system satisfies Galochkin's condition.*

5 Differential equations satisfied by G-functions

5.1 Introduction

Theorem 4.4.3 does not only give an interesting irrationality result for values of G-functions, it is also of fundamental importance for the theory of G-functions. We discuss the following consequence here.

Theorem 5.1.1 *Let f be a non-trivial G-function and let $Ly = 0$ be the linear differential equation of minimal order having f as a solution. Then $Ly = 0$ is a Fuchsian linear differential equation with rational local exponents.*

The usual proof goes roughly like this. Choodnovsky's theorem implies that the equation $Ly = 0$ is globally nilpotent, in the terminology of Dwork and Katz, see [11], p98. Then a Theoreom by Katz implies that $Ly = 0$ is Fuchsian with rational local exponents.
In these notes we shall adopt a more direct approach and a more modest goal.

Theorem 5.1.2 *Let f be a non-trivial G-function and let $Ly = 0$ be the linear differential equation of minimal order having f as a solution. Then $z = 0$ is either regular or a regular singular point of $Ly = 0$.*

5.2 Proof of Theorem 4.4.3

Let us adopt the notations of Theorem 4.4.3. Let $T(z)$ be the common denominator of the entries of G. We will assume that $T(z)$ and the entries of $T(z)G$ are polynomials with coefficients in the algebraic integers.
Let N, M be positive integers and let $Q, P_1, \ldots, P_n$ be polynomials of degree at most N, not all trivial, such that

$$Q\mathbf{f} - \mathbf{P} = O(z^{N+M})$$

where $\mathbf{P}$ stands for the column vector with entries $P_1, \ldots, P_n$. Take the derivative on both sides. We get

$$Q'\mathbf{f} + QG\mathbf{f} - D\mathbf{P} = O(z^{N+M-1}$$

where D denotes derivation with respect to z. Subtract G applied to the original equation to get

$$Q'\mathbf{f} - (D - G)\mathbf{P} = O(z^{N+M-1}).$$

Repeat this m times and divide $m!$ and multiply by $T(z)^m$,

$$\frac{T(z)^m}{m!} Q^{(m)}\mathbf{f} - \frac{T(z)^m}{m!}(D - G)^m \mathbf{P} = O(z^{N+M-m}).$$

Let t the maximum of the degrees of $T(z)$ and the entries of $T(z)G$. Suppose that $M - m > mt$. The idea of Choodnovsky is that the coefficients of the entries of $\frac{T(z)^m}{m!}(D - G)^m\mathbf{P}$ coincide with the Taylor series coefficients of $\frac{T(z)^m}{m!} Q^{(m)}\mathbf{f}$. At least, the coefficients of all powers z^k with $k < N + M - m$. Determination of the arithmetic nature of the coefficients of $\frac{T(z)^m}{m!} Q^{(m)}\mathbf{f}$ is easy. We can then use this information to deduce something about G_s.

Lemma 5.2.1 *Let G_s be the $n\times n$-matrix with entries in $\overline{\mathbb{Q}}(z)$ such that*

$$\frac{1}{s!}\mathbf{y}^{(s)}=G_s\mathbf{y}$$

for every solution $\mathbf{y}$ of the system $'y'=G\mathbf{y}$. Then, for any n-vector $\mathbf{P}$ we have

$$G_s\mathbf{P}=\sum_{j=0}^{s}\frac{(-1)^j}{(s-j)!j!}D^{s-j}(D-G)^j\mathbf{P}.$$

Proof. Induction on s. □

Let us write $\mathbf{P}_h=\frac{1}{h!}(D-G)^h\mathbf{P}$. Then it follows from our Lemma that

$$G_s\mathbf{P}_h=\sum_{j=0}^{s}\frac{(-1)^j}{(s-j)!}\binom{j+h}{h}D^{s-j}\mathbf{P}_{j+h}.$$

Let us now write $R_{(k)}$ for the $n\times n$ matrix with the columns $\binom{k+h}{h}P_{k+h}$ for $h=0,1,\ldots,n-1$. Then we obtain the $n\times n$ matrix equality.

$$G_sR_{(0)}=\sum_{j=0}^{s}\frac{(-1)^j}{(s-j)!}D^{s-j}R_{(j)}. \tag{4}$$

According to Shidlovskii's Lemma 6.2.7 the matrix $R_{(0)}$ is invertible. So we can express G_s in terms of the $\mathbf{P}_m$ with $m\le s+n-1$.
Let us suppose that the coefficients of the polynomial Q are algebraic integers of size bounded by C. Let D be the common denominator of the coefficients of $1,z,z^2,\ldots,z^{N+M}$ of the components of $\mathbf{f}$. Then the polynomial components of $\mathbf{P}$, defined by $DQ(z)\mathbf{f}-\mathbf{P}=O(z^{N+M})$ have algebraically integral coefficients. For any m we have the relation

$$\frac{T(z)^m}{m!}DQ^{(m)}(z)\mathbf{f}-T(z)^m\mathbf{P}_m=O(z^{N+M-m}).$$

Consequently the coefficients of the polynomial components of $T(z)^m\mathbf{P}_m$ are algebraically integral whenever $N+M-m>N+tm$. A fortiori the entries of $T(z)^mR_{(m)}$ are polynomials with algebraically integrals. Similarly the coefficients of the entries of $\frac{T(z)^j}{(s-j)!}D^{s-j}R_{(j)}$ are polynomial with algebraically integral coefficients. To estimate the common denominator of the coefficients of the entries of $T(z)^sG_s$ we need to estimate the size of the coefficients of $T(z)^{n-1}\det(R_{(0)})$. It is straightforward to check that this size is bounded by $c_1(DC)^n$, where c_1 is a constant depending only on the system $y'=Gy$ and not on Q.
Let us now set $M=N/(2n)$ and $s=[N/(2n(t+1))]$. Then the system $Q(z)\mathbf{f}-\mathbf{P}=O(z^{N+M})$ represents a system of $nN/(2n)$ linear homogeneous equations for the $N+1$ unknown coefficients of $Q(z)$. According to Siegel's Lemma there exists a solution $Q(z)$, non-trivial, whose coefficients are bounded in size by c_2D where c_2 depends only on $\mathbf{f}$. Since $\mathbf{f}$ has G-functions as components there exists $\gamma>0$, depending only on $\mathbf{f}$, such that $D<\gamma^{M+N}$. Hence the size of the coefficients of $\det(R_{(0)})$ is bounded by $c_1(CD)^n<c_1(c_2D^2)^n<c_1c_2^n(\gamma^{2Mn+2Nn})=c_1c_2^n\gamma^{(2n+1)N}$. This is also the bound for the common denominator of $G_1,G_2,\ldots,G_s$. In terms of s this bound reads $c_1c_2^n\gamma^{2n(2n+1)(t+1)s}$. In other words, an exponential bound in s and Choodnosvky' theorem is proved.

5.3 Proof of Theorem 5.1.2

In this section we start with the differential equation $Ly = 0$ which we assume has order n. Consider the corresponding system of first order equations. By Theorem 4.4.3 this system satisfies the Galochkin condition. Let us see what this means in terms of the differential equation. Write the differential $Ly = 0$ equation as

$$T(z)y^{(n)} = Q_{n-1}(z)y^{(n-1)} + \cdots + Q_1(z)y' + Q_0 y$$

where $T(z), Q_0(z), \ldots, Q_{n-1}(z)$ are polynomials in $\overline{\mathbb{Q}}[z]$ with algebraically integral coefficients. By recursion on m we find rational functions $Q_{m,r} \in \overline{\mathbb{Q}}(z)$ for $r = 0, 1, \ldots, n-1$ such that

$$y^{(m)} = Q_{m,n-1}(z)y^{(n-1)} + \cdots + Q_{m,1}(z)y' + Q_{m,0}(z)y.$$

In particular $Q_{n,r}(z) = Q_r(z)/T(z)$. By construction $T(z)^{m-n+1}Q_{m,r}$ is a polynomial in $\overline{\mathbb{Q}}[z]$ with algebraically coefficients for every m and $r = 0, 1, \ldots, n-1$.
Galoschkin's condition now comes down to the fact that there exists $C > 0$ such that for every integer N the common denominator of all coefficients of all polynomials $\frac{1}{m!}T(z)^{m-n+1}Q_{m,r}$ with $n \le m \le N, 0 \le r \le n-1$ is bounded by C^N.
The proof of Theorem 5.1.2 can be achieved by using Theorem 4.4.3 and the following Proposition.

Proposition 5.3.1 *Let $Ly = 0$ be a linear differential equation which satisfies Galoschkin's condition. Then $z = 0$ is a regular singularity.*

Proof. Let us write the equation $Ly = 0$ in the form

$$z^n y^{(n)}(z) = B_{n-1}(z)z^{n-1}y^{(n-1)}(z) + \cdots + B_1(z)zy'(z) + B_0(z)y(z).$$

Here the $B_k(z)$ are rational functions. Suppose their common denominator is $T(z)$. When the $B_k(z)$ are all regular in $z = 0$ the point $z = 0$ is a regular singularity. Define

$$\lambda = \max(v(B_{n-1}), \frac{1}{2}v(B_{n-2}), \frac{1}{3}v(B_{n-3}), \ldots, \frac{1}{n}v(B_0))$$

where $v(R)$ denotes the pole order of $R(z)$ at $z = 0$. Then $z = 0$ is an irregular singularity if and only if $\lambda > 0$. Let us suppose that $\lambda > 0$. For any m we define the rational functions $A_{m,r}$ by

$$z^m y^{(m)} = \sum_{r=0}^{n-1} A_{m,r}(z)z^r y^{(r)}. \tag{5}$$

We will show that the limit of $z^{(m-r)\lambda}A_{m,r}$ exists as $z \to 0$. Call this limit $\alpha_{m,r}$. We will also show that $\alpha_{m,r}$ is non-zero infinitely often and grows at most exponentially in m. As a result our equation cannot satisfy Galochkin's condition.
We first derive a recurrence relation for the $A_{m,r}$. Take the derivative of (5) and multiply by z,

$$mz^m y^{(m)} + z^{m+1}y^{(m+1)} = \sum_{r=0}^{n-1}(zA'_{m,r} + rA_{m,r})z^r y^{(r)} + A_{m,r}z^{r+1}y^{(r+1)}.$$

Apply the differential equation to change the right hand side into

$$\sum_{r=0}^{n-1}(zA'_{m,r}+rA_{m,r})z^r y^{(r)} + A_{m,n-1}\left(\sum_{r=0}^{n-1} B_r z^r y^{(r)}\right) + \sum_{r=1}^{n-1} A_{m,r-1}z^r y^{(r)}.$$

Adopting the convention $A_{m,-1}=0$ this equals

$$\sum_{r=0}^{n-1}(zA'_{m,r}+rA_{m,r}+B_rA_{m,n-1}+A_{m,r-1})z^r y^{(r)}.$$

As a result we derive the following recurrence relation,

$$A_{m+1,r} = zA'_{m,r} + (r-m)A_{m,r} + B_rA_{m,n-1} + A_{m,r-1}.$$

Define $\tilde{A}_{m,r} = z^{(m-r)\lambda}A_{m,r}$ and $\tilde{B}_r = z^{(n-r)\lambda}B_r$. After multiplication by $z^{(m+1-r)\lambda}$ the recurrence changes into

$$\tilde{A}_{m+1,r} = z^{\lambda+1}\tilde{A}'_{m,r} + (r-m)(1+\lambda)z^\lambda \tilde{A}_{m,r} + \tilde{B}_r\tilde{A}_{m,n-1} + \tilde{A}_{m,r-1}.$$

We define $\beta_r = \lim_{z\to 0}\tilde{B}_r$. Notice that this limit exists by the definition of λ and that not all β_r are zero. We now let $z\to 0$ in our recurrence and define $\alpha_{m,r} = \lim_{z\to 0}\tilde{A}_{m,r}$. We obtain

$$\alpha_{m+1,r} = \beta_r\alpha_{m,n-1} + \alpha_{m,r-1}.$$

In matrix form

$$\begin{pmatrix}\alpha_{m+1,n-1}\\ \vdots\\ \alpha_{m+1,1}\\ \alpha_{m+1,0}\end{pmatrix} = \begin{pmatrix}\beta_{n-1} & 1 & 0 & \dots & 0\\ \beta_{n-2} & 0 & 1 & \dots & 0\\ \vdots & & & \ddots & 0\\ \beta_1 & 0 & 0 & \dots & 1\\ \beta_0 & 0 & 0 & \dots & 0\end{pmatrix}\begin{pmatrix}\alpha_{m,n-1}\\ \vdots\\ \alpha_{m,1}\\ \alpha_{m,0}\end{pmatrix}.$$

It follows from the definitions that $\alpha_{n,r}=\beta_r$. Writing

$$\mathcal{B} = \begin{pmatrix}\beta_{n-1} & 1 & 0 & \dots & 0\\ \beta_{n-2} & 0 & 1 & \dots & 0\\ \vdots & & & \ddots & 0\\ \beta_1 & 0 & 0 & \dots & 1\\ \beta_0 & 0 & 0 & \dots & 0\end{pmatrix}$$

it is not hard to verify that

$$\begin{pmatrix}\alpha_{m,n-1}\\ \vdots\\ \alpha_{m,1}\\ \alpha_{m,0}\end{pmatrix} = \mathcal{B}^m\begin{pmatrix}0\\ \vdots\\ 0\\ 1\end{pmatrix}.$$

The latter vector is non-zero for every m and its coefficients have size which grows at most exponentially in m.

5.4 Proof of Theorem 3.4.1

We are now ready to prove André's theorem on E-functions. Notice that if $f(z) = \sum_{n\ge0} \frac{a_n}{n!} z^n$ is an E-function then, by definition, $g(z) = \sum_{n\ge0} a_n z^n$ is a G-function. They are related by the formal Laplace transform

$$\int_0^\infty e^{-xz} f(z)dz = \frac{1}{x} g(\frac{1}{x}).$$

For any non-negative integers k, m and repeated partial integration we can derive the equality

$$\int_0^\infty e^{-xz} \left(\frac{d}{dz}\right)^k z^m f(z)dz = x^k \left(-\frac{d}{dx}\right)^m \frac{1}{x} g(\frac{1}{x}).$$

Now let us assume that $g(x)$ is a G-function and so, it satisfies a Fuchsian differential equation. Hence $\frac{1}{x}g(\frac{1}{x})$ also satisfies a linear differential equation. Assume that it is of the form

$$\sum_{k=0}^K \sum_{m=0}^M A_{k,m} x^k \left(-\frac{d}{dx}\right)^m \frac{1}{x} g(\frac{1}{x}) = 0.$$

Then, by our Laplace transform property,

$$0 = \int_0^\infty e^{-xz} \sum_{k=0}^K \sum_{m=0}^M A_{k,m} \left(\frac{d}{dz}\right)^k z^m f(z)dz.$$

Hence

$$\sum_{k=0}^K \sum_{m=0}^M A_{k,m} \left(\frac{d}{dz}\right)^k z^m f(z)dz = 0.$$

We assume that K is the largest index such that $A_{K,m} \ne 0$ for some m. Assume that M is the largest index such that $A_{k,M} \ne 0$ for some k. So g satisfies an equation of order M and f is annihilated by an operator of order K. Because we are dealing with a Fuchsian operator we have that $A_{K,M} \ne 0$ and $A_{K,m} = 0$ for all $m < M$. For the operator annihiliting f this means that it is of order K and that the coefficient of $f^{(K)}(z)$ equals $A_{K,M} z^M$. This is precisely the content of Theorem 3.4.1.

6 Siegel's method

In this section we sketch the ideas of Siegel and Shidlovskii to obtain transcendence statements and irrationality results for values of E-functions and G-functions.

6.1 Siegel's Lemma

Theorem 6.1.1 (Siegel's Lemma) *Let K be a number field of degree δ. Consider the system of linear equations*

$$\sum_{j=1}^{n} a_{ij}x_j = 0 \quad (i = 1, \dots, m)$$

where $a_{ij} \in \mathcal{O}_K$ for all i, j. Let

$$A = \max_{ij} \overline{|a_{ij}|}.$$

Suppose $m < n$. Then the system has a non-trivial solution in algebraic integers x_j in K such that

$$\max_i \overline{|x_i|} \leq c_1(c_1 nA)^{m/(n-m)}.$$

Here c_1 is positive number depending only on K.

6.2 Shidlovskii's Lemma

In his original work Siegel had to impose a number of complicated technical conditions on the system of differential equations. This was circumvented in an elegant way by A.B.Shidlovskii in 1956. The main idea rests on the following Proposition.

Proposition 6.2.1 *Let A be an $n \times n$ matrix with entries in $\mathbb{C}(z)$. Denote for any column vector $\mathbf{P}$ with entries in $\mathbb{C}(z)$, $(D - A)\mathbf{P} = \mathbf{P}' - A\mathbf{P}$.*
Then there exists a constant $c(A) > 0$ depending only on A such that for any non-trivial $\mathbf{P} \in \mathbb{C}(z)^n$ the $\mathbb{C}(z)$-linear space spanned by $(D - A)^i\mathbf{P}$, $\quad i = 0, 1, 2, \dots$ has a basis consisting of vectors in $\mathbb{C}(z)^n$ all of whose non-zero components have degrees bounded by $c(A)$.

Proof. Suppose that the dimension of the space spanned by all $(D - A)^i\mathbf{P}$ is r. Then the vectors $(D - A)^i\mathbf{P}$ with $i = 0, 1, 2, \dots, r - 1$ form a basis of this space. Moreover there exist rational functions $A_0, A_1, \dots, A_{r-1}$ such that

$$(D - A)^r\mathbf{P} = A_{r-1}(D - A)^{r-1}\mathbf{P} + \cdots + A_1(D - A)\mathbf{P} + A_0\mathbf{P}.$$

Let $\mathbf{y}$ be any solution of the system $\mathbf{y}' = -A^t\mathbf{y}$ where A^t denotes the transpose of A. A straightforward computation shows that $(D - A)^i\mathbf{P} \cdot \mathbf{y} = (\mathbf{P} \cdots \mathbf{y})^{(i)}$ for any i. Furthermore $\mathbf{P} \cdot \mathbf{y}$ satisfies the linear differential equation

$$u^{(r)} = A_{r-1}u^{(r-1)} + \cdots + A_1u' + A_0u.$$

Denote the $\mathbb{C}$-linear space of solutions of this equation by W and the space of solutions of the system $\mathbf{y}' = -A^t\mathbf{y}$ by V. Then the map $\Phi : \mathbf{y} \mapsto \mathbf{P}\cdot\mathbf{y}$ defines a $\mathbb{C}$-linear map from V to W. Since $\dim(W) = r$ and $\dim(V) = n$ the kernel of Φ has dimension at least $n-r$. Moreover, $(D-A)^i\mathbf{P}\cdot\mathbf{y} = (\mathbf{P}\cdot\mathbf{y})^{(i)} = 0$ for any $\mathbf{y} \in \ker(\Phi)$ and $i = 0, 1, \ldots, r-1$. Hence $\dim\ker(\Phi)$ is at most $n-r$. Therefore the dimension equals $n-r$.
The space $\ker(\Phi)$ is now a differentially invariant subspace (see Definition 6.2.2) of solutions of $\mathbf{y}' = -A\mathbf{y}$. The space of relation between the components of all $\mathbf{y} \in \ker(\Phi)$ is precisely the space spanned by the $(D-A)^i\mathbf{P}$. Hence application of Lemma 6.2.3 with $-A^t$ instead of A gives us the desired basis. □

Definition 6.2.2 *Consider an $n \times n$ system of first order linear equations. A linear subspace W of the vector space of solutions is called differentially invariant if the space of vectors $\mathbf{Q} \in \mathbb{C}(z)^n$ such that $\mathbf{Q}\cdot\mathbf{y} = 0$ for all $\mathbf{y} \in W$ has $\mathbb{C}(z)$-dimension $n - \dim(W)$.*

Clearly, in the proof of Shidlovskii's Lemma we encounter such subspaces.

Lemma 6.2.3 *Consider the $n\times$ n-system of differential equations $\mathbf{y}' = A(z)\mathbf{y}$. Then there exists a constant $c(A)$, depending only on A, with the following property. Let W be any differentially invariant subspace of solutions of $\mathbf{y}' = A\mathbf{y}$ and suppose its dimension is s. Then there is a basis $\mathbf{Q}_1, \ldots, \mathbf{Q}_{n-s}$ for the $\mathbb{C}(z)$-linear relations between the components of $\mathbf{y} \in W$ all of whose entries have degree $\le c(A)$.*

Proof. Choose a basis $\mathbf{y}_1, \ldots, \mathbf{y}_s$ of W. Let Y be the $n \times s$ matrix whose columns are the vectors $\mathbf{y}_i$ for $i = 1, 2\ldots, s$. Without loss of generality we can assume that the first s rows of Y have rank s. Denote the entries of Y by Y_{ij} and let $V = (Y_{ij})_{i,j=1,\ldots,s}$. A basis for the $\mathbb{C}(z)$-linear relations can be obtained as follows. For any $h = 1, 2, \ldots, n-s$ the $h+s$-th row of Y can be written as a linear combination of the first s rows with coefficients $A_1, \ldots, A_s$ say. Since the space of $\mathbb{C}(z)$-linear relations has rank $n-s$ we have $A_i \in \mathbb{C}(z)$ for $i = 1, \ldots, s$. By Cramer's rule every A_i is the quotient of $s \times s$ submatrices of Y.
The $s \times s$ submatrices of Y form the components of an $\binom{n}{s} \times \binom{n}{s}$ system of first order linear differential eqations which is called the s-th exterior power of $\mathbf{y}' = A\mathbf{y}$. We denote it by $\mathbf{u}' = \mathcal{A}^{(s)}\mathbf{u}$. Application of Lemma 6.2.4 to the latter system shows that the coefficients A_i have a degree bounded only dependent on s, and not on W itself. □

Lemma 6.2.4 *Consider an $n \times n$ system of first order linear differential equations. There exists a constant $c(A) > 0$ with the following property. Suppose that the components of a vector solution $\mathbf{y}$ have a ratio $R(z) \in \mathbb{C}(z)$. Then the degree of $R(z)$ is bounded by $c(A)$.*

Proof. Suppose the components involved are the first and second component Let $\mathbf{y}_1, \ldots, \mathbf{y}_n$ be a basis of solutions of the system. Denote $\mathbf{y}_i = (y_{i1}, y_{i2}, \ldots, y_{in})$. There exist $\lambda_1, \ldots, \lambda_n$ such that

$$R(z) = \frac{\lambda_1 y_{11} + \lambda_2 y_{21} + \cdots \lambda_n y_{n1}}{\lambda_1 y_{12} + \lambda_2 y_{22} + \cdots \lambda_n y_{n2} \in \mathbb{C}(z)}.$$

Let $v_1, \ldots, v_m$ be a basis of the $\mathbb{C}(z)$-linear vector space spanned by all y_{i1}, y_{i2}. Suppose $y_{i1} = \sum_{k=1}^m \mu_{i,k} v_k$ and $y_{i2} = \sum_{k=1}^m \nu i, k v_k$. Then, concentrating on the coefficients of v_1,

$$R(z) = \frac{\lambda_1\mu_{11} + \lambda_2\mu_{21} + \cdots + \lambda_n\mu_{n1}}{\lambda_1\nu_{11} + \lambda_2\nu_{21} + \cdots + \lambda_n\nu_{n1}}.$$

Since the $\mu i1, \nu_{i1}$ are independent of the choice of particular solution vector, we see that the degree of $R(z)$ is bounded by a number depending only on A. □

Lemma 6.2.5 *Suppose we have n functions $f_1, \ldots, f_n$ analytic around $z = 0$ and $\mathbb{C}(z)$-linearly independent. Let M be a positive integer and let $R_1, \ldots, R_n \in \mathbb{C}(z)$ be rational functions of degree bounded above by M. Then the vanishing order of the linear combination*

$$R_1 f_1 + R_2 f_2 + \cdots + R_n f_n$$

is bounded above by a number depending only on the $\mathbf{f}_j$ and M.

Proof. Exercise.

We shall now formulate and prove the two version of Shidlovskii's Lemma.
Let $\mathbf{f}(z)$ be a vector solution of an $n \times n$-system of first order linear differential equations

$$\frac{d\mathbf{f}}{dz} = A(z)\mathbf{f}.$$

Suppose that the entries of $A(z)$ are rational functions in $\mathbb{C}(z)$ and that the components of $\mathbf{f}$ are Taylor series in z. We denote the common denominator of the entries of A by $T(z)$. Consider a linear form $\mathbf{P} \cdot \mathbf{f} = P_1 f_1 + P_2 f_2 + \cdots + P_n f_n$ where the $P_i \in \mathbb{Z}[z]$ and $f_1, \ldots, f_n$ are the components of $\mathbf{f}$. From such a linear form we can produce a derived linear form by differentiation. Namely

$$(\mathbf{P} \cdot \mathbf{f})' = \mathbf{P}' \cdot \mathbf{f} + \mathbf{P} \cdot \mathbf{f}' = \mathbf{P}' \cdot \mathbf{f} + (A^t \mathbf{P}) \cdot \mathbf{f},$$

where $\mathbf{P}$ denotes the column vector with components $(P_1, \ldots, P_n)$ and A^t is the transpose of A. The newly formed linear form has coefficient vector $\mathbf{P'} + A^t\mathbf{P}$. Let us denote it by $(D + A^t)P$. Now form the $n \times n$ coefficient determinant Δ of the vectors $\mathbf{P}, (D + A^t)\mathbf{P}, (D + A^t)^2\mathbf{P}, \ldots, (D + A^t)^{n-1}\mathbf{P}$. In the Siegel-Shidlovskii method it is of crucial importance that $\Delta(z)$ is non-vanishing.

Theorem 6.2.6 (Shidlovskii's Lemma I) *Let $0 < \epsilon < 1$ and let N be a positive integer. Let $P_1, \ldots, P_n$ be polynomials, not all trivial, of degrees $\le N$ such that $\mathbf{P} \cdot \mathbf{f} = P_1 f_1 + \ldots + P_n f_n$ has vanishing order at least $(n - \epsilon)N$. Then, when N is sufficiently large, $\Delta(z) \ne 0$.*

Proof. Consider the $\mathbb{C}(z)$-linear space spanned by $(D+A^t)^i\mathbf{P}$ and suppose its dimension is r. According to Proposition 6.2.1 there exists $c_1(A) > 0$ depending only on A, and a $\mathbb{C}(z)$-basis $\mathbf{R}_1, \ldots, \mathbf{R}_r$ of the space where the degree of the entries of all $\mathbf{R}_i$ is bounded by $c(A)$. Let $T(z)$ be a common denominator for the entries of $A(z)$. Then the polynomial vectors $T(z)^k(D+A^t)^k\mathbf{P}$ are $\mathbb{C}(z)$-linear combinations of $\mathbf{R}_1,\ldots,\mathbf{R}_r$ for $k = 0,1,\ldots,r-1$. Let t be the maximum of the degree of $zT(z)$ and the degrees of the entries of $T(z)A(z)$. Then an upper bound for the degree of the entries of $z^kT(z)^k(D+A^t)^k\mathbf{P}$ is $N+kt$. Furthermore, $z^kT(z)^k(D+A^t)^k\mathbf{P}\cdot\mathbf{f} = O(z^{(n-\epsilon)N})$. As a result, at least one of the linear forms $\mathbf{R}_l\cdot\mathbf{f}$ has vanishing order

$$(n-\epsilon)N - r(N+(r-1)t) > (n-r-\epsilon)N - r^2t.$$

According to Lemma 6.2.5 all linear forms $\mathbf{R}_l\cdot\mathbf{f}$ have vanishing order at most $c_2(A)$, depending only on A. Therefore, whenever $N > (r^2t + c_2(A))/(1-\epsilon)$ and $n-r>0$ we get a contradiction. Hence $r=n$ if N is sufficiently large. □

The second version of Shidlovskii's Lemma deals with simultaneous rational approximations to the components of a Taylor series solution of a first order system.
Consider again the system $\mathbf{y}' = A\mathbf{y}$ and let $\mathbf{f}$ be a solution vector whose components are Taylor series.
Let N,M be positive integers and let $Q,P_1,\ldots,P_n$ be polynomials of degree at most N, not all trivial, such that

$$Q\mathbf{f} - \mathbf{P} = O(z^{N+M})$$

where $\mathbf{P}$ stands for the column vector with entries $P_1,\ldots,P_n$. Take the derivative on both sides. We get

$$Q'\mathbf{f} + QA\mathbf{f} - D\mathbf{P} = O(z^{N+M-1}$$

where D denotes derivation with respect to z. Subtract A applied to the original equation to get

$$Q'\mathbf{f} - (D-A)\mathbf{P} = O(z^{N+M-1}).$$

Repeat this m times,

$$Q^{(m)}\mathbf{f} - (D-A)^m\mathbf{P} = O(z^{N+M-m}).$$

The determinant of the vectors $(D-A)^m\mathbf{P}$, $\quad m = 0,1,\ldots,n-1$ is denoted by $\Delta(z)$.

Theorem 6.2.7 (Shidlovskii's Lemma II) *Let $0<\epsilon<1$ and let notations be as above. Then, when $M > N(1-\epsilon)/n$ and N is sufficiently large, $\Delta(z)\ne 0$.*

Proof. To be added...

6.3 Siegel's method for E-functions, sketch

Suppose that $\mathbf{f}$ is an E-function solution of a system

$$\mathbf{f}'(z) = A(z)\mathbf{f}(z)$$

where the entries of $A(z)$ are in $\overline{\mathbb{Q}}(z)$. We assume that $f_1,\ldots,f_n$ are $\overline{\mathbb{Q}}(z)$-linear independent. Fix $0<\theta<1$ and choose N. We construct polynomials $P_1,\ldots,P_n$ of degree $\le N$ such that

$$P_1f_1 + \cdots + P_nf_n = O(z^{(n-\theta)N}).$$

This requires the solution of $(n-\theta)N$ homogeneous linear equations in the $n(N+1)$ unknown coefficients of the P_i. In other words, the number of variables exceeds the number of equations and we can invoke Siegel's Lemma to contruct polynomials P_i whose coefficients are bounded in terms of N. We summarise with the following Lemma.

Lemma 6.3.1 *For any sufficiently large integer N there are polynomials $P_i(z), i = 1,\ldots,n$, not all identically zero with degrees at most N and algebraic integer coefficients of size at most $(N!)^{1+\theta}$ such that*

$$P_1(z)f_1(z) + \cdots + P_n(z)f_n(z) = \sum_{m\geq(n-\theta)N} \rho_m z^m$$

where $|\rho_m| < N!(m!)^{-1+\theta}$.

To be finished...

7 References

References

[1] Y. André, *G-functions and Geometry*, Aspects of Mathematics, Vieweg, 1989

[2] Y. André, Gevrey series of arithmetic type I, Purity and duality theorems. Ann. Math. 151 (2000), 705-740.

[3] Y. André, Gevrey series of arithmetic type II, Transcendence without transcendence. Ann. Math. 151 (2000), 741-756.

[4] Y. André, Arithmetic Gevrey series and transcendence, a survey, J.Théorie des Nombres de Bordeaux 15(2003), 1-10.

[5] K. Ball, T. Rivoal, Irrationalit d'une infinit de valeurs de la fonction zta aux entiers impairs. (French) [Irrationality of infinitely many values of the zeta function at odd integers] Invent. Math. 146 (2001), 193–207.

[6] F. Beukers, A note on the irrationality of $\zeta(3)$, Bull.London Math.Soc 11 (1979), 268-272.

[7] F. Beukers, W.D. Brownawell, G. Heckman, *Siegel normality*, Annals of Math 127 (1988), 279-308.

[8] F. Beukers, A refined version of the Siegel-Shidlovskii theorem, Ann. Math. 163 (2006), 369-379

[9] E. Bombieri, On G-functions, *Recent Progress in analytic Number Theory*, eds. H.Halberstamm and C.Hooley, Academic Press, 1981. Vol 2, pp 1-67.

[10] G.V. Choodnovsky, On applications of diophantine approximations, Proc.Nat.Acad.Sci. 81(1984), 7261-7265.

[11] B. Dwork, G. Gerotto, F.J. Sullivan, *An introduction to G-functions*, Annals of Math Studies 133, Princeton Univ. Press 1994.

[12] B. Edixhoven, A. Yafaev, Subvarieties of Shimura varieties. Ann. of Math. 157 (2003), 621–645.

[13] A.I. Galochkin, Estimates from below of polynomials in the values of analytic functions of a certain class, Math USSR Sbornik 24(1974), 385-407., original article in Mat.Sbornik 95(137)(1974).

[14] M. Kontsevich, D. Zagier, Periods, in *Mathematics unlimited—2001 and beyond*, 771–808, Springer, Berlin, 2001.

[15] Yu.V.Nesterenko, A.B.Shidlovskii, On the linear independence of values of E- functions, Math.Sb 187(1996), 93-108, translated in Sb.Math 187 (1996), 1197- 1211.

[16] A.J. van der Poorten, A proof that Euler missed... Apé ry's proof of the irrationality of $\zeta(3)$, Math Intelligencer 1 (1978), p. 195-203.

[17] C.L. Siegel, Über einige Anwendungen diophantischer Approximationen, Abh. der Preussischen Akademie der Wissenschaften 1 (1929), *Gesammelte Abhandlungen*, Band I pp 208-266, Springer-Verlag 1966.

[18] C.L. Siegel, *Transcendental Numbers*, Annals of Math Studies 16, Princeton Univ. Press 1949.

[19] A.B. Shidlovskii, *Transcendental Numbers*, W.de Gruyter, Berlin, New York, 1989

[20] A.B. Shidlovskii, A criterion for algebraic independence of the values of a class of entire functions. (Russian) Izv. Akad. Nauk SSSR. Ser. Mat. 23 (1959), 35–66

[21] J.Wolfart, Werte hypergeometrischer Funktionen, Invent. Math. 92 (1988), 187-216