Historical introduction to irrationality

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Abstract

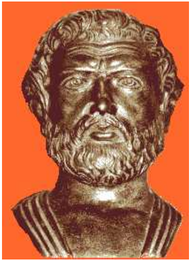
This first lecture deals with irrationality, as an introduction to transcendence results.

After a short historical survey of this topic, we reproduce the easy proof of irrationality of e by Fourier (1815) and explain how Liouville extended it up to a proof that e^2 is not a quadratic number.

Such irrationality proofs rest on a criterion for irrationality, which can be extended to a criterion of linear independence.

Irrationality of $\sqrt{2}$

Pythagoreas school



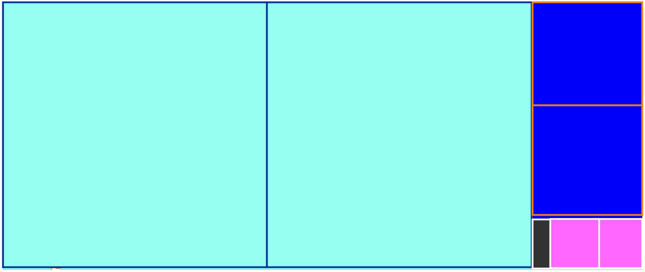
Hippasus of Metapontum (around 500 BC).

Sulba Sutras, Vedic civilization in India, ∼800-500 BC.

Irrationality of $\sqrt{2}$: geometric proof

- Start with a rectangle have side length 1 and $1+\sqrt{2}$.
- Decompose it into two squares with sides 1 and a smaller rectangle of sides $1+\sqrt{2}-2=\sqrt{2}-1$ and 1.
- This second small rectangle has side lenghts in the proportion
$$\frac{1}{\sqrt{2}-1}=1+\sqrt{2},$$
which is the same as for the large one.
- Hence the second small rectangle can be split into two squares and a third smaller rectangle, the sides of which are again in the same proportion.
- This process does not end.

Rectangles with proportion $1 + \sqrt{2}$



Irrationality of $\sqrt{2}$: geometric proof

If we start with a rectangle having integer side lengths, then this process stops after finitely may steps (the side lengths are positive decreasing integers).

Also for a rectangle with side lengths in a rational proportion, this process stops after finitely may steps (reduce to a common denominator and scale).

Hence $1+\sqrt{2}$ is an irrational number, and $\sqrt{2}$ also.

Continued fraction

The number

$$\sqrt{2} = 1.414\,213\,562\,373\,095\,048\,801\,688\,724\,209\ \ldots$$

satisfies

$$\sqrt{2} = 1 + \frac{1}{\sqrt{2}+1}\cdot$$

Hence

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{\sqrt{2}+1}} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}$$

We write the continued fraction expansion of $\sqrt{2}$ using the shorter notation

$$\sqrt{2} = [1;\ 2,\ 2,\ 2,\ 2,\ 2,\ \ldots] = [1;\ \overline{2}].$$

Pell's equation :

$$x^2 - dy^2 = \pm 1$$

• H.W. Lenstra Jr,
Solving the Pell Equation,
Notices of the A.M.S.
49 (2) (2002) 182–192.

Irrationality criteria

A real number is rational if and only if its continued fraction expansion is finite.

A real number is rational if and only if its binary (or decimal, or in any basis $b \geq 2$) expansion is *ultimately periodic*.

Consequence : it should not be so difficult to decide whether a given number is rational or not.

To prove that certain numbers (occurring as constants in analysis) are irrational is most often an impossible challenge. However to construct irrational (even transcendental) numbers is easy.

Euler–Mascheroni constant

Euler's Constant is

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right)$$
$$= 0.577\,215\,664\,901\,532\,860\,606\,512\,090\,082\ldots$$

Is–it a rational number ?

$$\gamma = \sum_{k=1}^{\infty}\left(\frac{1}{k}-\log\left(1+\frac{1}{k}\right)\right) = \int_1^\infty\left(\frac{1}{[x]}-\frac{1}{x}\right)dx$$
$$= -\int_0^1\int_0^1 \frac{(1-x)dxdy}{(1-xy)\log(xy)}.$$

Recent work by *J. Sondow* inspired by the work of F. Beukers on Apéry's proof.

Riemann zeta function

The function
$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$
was studied by Euler (1707– 1783)
for integer values of s
and by Riemann (1859) for complex values of s.

Euler : for any even integer value of $s \geq 2$, the number $\zeta(s)$ is a rational multiple of π^s.

Examples : $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$, $\zeta(6) = \pi^6/945$, $\zeta(8) = \pi^8/9450 \cdots$

Coefficients : Bernoulli numbers.

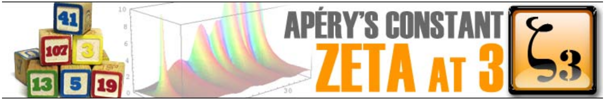
Introductio in analysin infinitorum

Leonhard Euler

(**15 Avril 1707** – 1783)

Introductio in analysin infinitorum

Riemann zeta function



The number

$$\zeta(3) = \sum_{n \geq 1} \frac{1}{n^3} = 1,202\,056\,903\,159\,594\,285\,399\,738\,161\,511\,\ldots$$

is irrational *(Apéry 1978)*.

Recall that $\zeta(s)/\pi^s$ is rational for any even value of $s \geq 2$.

Open question : Is the number $\zeta(3)/\pi^3$ irrational ?

Is the number

$$\zeta(5) = \sum_{n\geq 1} \frac{1}{n^5} = 1.036\,927\,755\,143\,369\,926\,331\,365\,486\,457\ldots$$

irrational ?

T. Rivoal (2000) : infinitely many $\zeta(2n+1)$ are irrational.

Tanguy Rivoal

Let $\epsilon > 0$*. For any sufficiently large odd integer* a*, the dimension of the* $\mathbf{Q}$*-vector space spanned by the numbers* $1,\ \zeta(3),\ \zeta(5),\ \cdots, \zeta(a)$ *is at least*

$$\frac{1-\epsilon}{1+\log 2}\log a.$$

Wadim Zudilin

- *One at least of the four numbers*
  $\zeta(5), \quad \zeta(7), \quad \zeta(9), \quad \zeta(11)$
  is irrational

- *There is an odd integer* j *in the interval* $[5, 69]$ *such that the three numbers*
  $1, \zeta(3), \zeta(j)$
  are linearly independent over $\mathbf{Q}$.

Open problems (irrationality)

- Is the number

$$e + \pi = 5.859\,874\,482\,048\,838\,473\,822\,930\,854\,632\ldots$$

  irrational ?
- Is the number

$$e\pi = 8.539\,734\,222\,673\,567\,065\,463\,550\,869\,546\ldots$$

  irrational ?
- Is the number

$$\log \pi = 1.144\,729\,885\,849\,400\,174\,143\,427\,351\,353\ \ldots$$

  irrational ?

Catalan's constant

Is Catalan's constant
$$\sum_{n\geq 1} \frac{(-1)^n}{(2n+1)^2}$$
$= 0.915\,965\,594\,177\,219\,015\,0\ldots$

an irrational number ?

This is the value at $s = 2$ of the Dirichlet L–function $L(s, \chi_{-4})$ associated with the Kronecker character
$$\chi_{-4}(n) = \left(\frac{n}{4}\right),$$
which is the quotient of the Dedekind zeta function of $\mathbf{Q}(i)$ and the Riemann zeta function.

Euler Gamma function

Is the number

$$\Gamma(1/5) = 4.590\,843\,711\,998\,803\,053\,204\,758\,275\,929\,152\ \ldots$$

irrational ?

$$\Gamma(z) = e^{-\gamma z} z^{-1} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n} = \int_0^{\infty} e^{-t} t^z \cdot \frac{dt}{t}$$

Here is the set of rational values for z for which the answer is known (and, for these arguments, the Gamma value is a transcendental number) :

$$r \in \left\{ \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6} \right\} \pmod 1.$$

Aryabhata I (476 – 550)

चतुरधिकं शतमष्टगुणं द्वाषष्टिस्तथा सहस्राणाम् ।
अयुतद्वयविष्कम्भस्यासन्नो वृत्तपरिणाहः ॥

100 plus 4, multiplied by 8, and added to 62,000: this is approximately the measure of the circumference of a circle of diameter 20,000.

This gives:

$$\pi = \frac{circumference}{diameter} = \frac{62832}{20000} = 3.1416\,,$$

seen to be accurate to four decimal places.

Known results

Irrationality of the number π :

Āryabhaṭa, b. 476 AD : $\pi \sim 3.1416$.

Nīlakaṇṭha Somayājī, b. 1444 AD : *Why then has an approximate value been mentioned here leaving behind the actual value ? Because it (exact value) cannot be expressed.*

K. Ramasubramanian, *The Notion of Proof in Indian Science*, 13th World Sanskrit Conference, 2006.

Irrationality of π

Johann Heinrich Lambert (1728 - 1777)
Mémoire sur quelques propriétés remarquables des quantités transcendantes circulaires et logarithmiques,
Mémoires de l'Académie des Sciences de Berlin, **17** (1761), p. 265-322 ;
read in 1767 ; Math. Werke, t. II.

$\tan(v)$ is irrational for any rational value of $v \neq 0$
and $\tan(\pi/4) = 1$.

Pi story

Frédéric II, King of Prussia
and H. Lambert
— What do you known, Lambert
— Everything, Majesty.
— And who taught you ?
— Myself !

Continued fraction expansion of $\tan(x)$

$$\tan(x) = \frac{1}{i}\tanh(ix), \qquad \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}\cdot$$

$$\tan(x) = \cfrac{x}{1 - \cfrac{x^2}{3 - \cfrac{x^2}{5 - \cfrac{x^2}{7 - \cfrac{x^2}{9 - \cfrac{x^2}{\ddots}}}}}}\cdot$$

$$\tan x = \frac{x|}{|1} - \frac{x^2|}{|\,3} - \frac{x^2|}{|\,5} - \frac{x^2|}{|\,7} - \cdots - \frac{x^2\quad|}{|2n+1} - \cdots$$

Continued fraction expansion

$$\frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{x|}{|1} + \frac{x^2|}{|\,3} + \frac{x^2|}{|\,5} + \frac{x^2|}{|\,7} + \cdots + \frac{x^2\quad|}{|2n+1} + \cdots$$

Define inductively

$$A_{n+1}(x) = (2n+1)A_n(x) + x^2 A_{n-1}(x) \quad (n \geq 1)$$

Let $u_n(x) = -A_n(x)/A_{n-1}(x)$. Then

$$u_n = \frac{-A_n}{A_{n-1}} = \frac{x^2 A_n}{(2n+1)A_n - A_{n+1}} = \frac{x^2}{2n+1-u_{n+1}}.$$

Continued fraction expansion (continued)

Therefore, for $k \geq 1$,

$$u_n = \frac{x^2 \;\big|}{\big| 2n+1} - \frac{x^2 \;\big|}{\big| 2n+3} - \cdots - \frac{x^2 \;\big|}{\big| 2n+2k+1} - u_{n+k+1}.$$

With the initial values

$$A_0(x) = e^x - 1, \quad A_1(x) = e^x(2-x) - 2 - x$$

the solution is

$$A_n(x) = \frac{x^{2n+1}}{n!} \int_0^1 e^{-tx} t^n (1-t)' dt.$$

S.A. Shirali – *Continued fraction for e*,
Resonance, vol. **5** N°1, Jan. 2000, 14–28.
http ://www.ias.ac.in/resonance/

Irregular continued fractions

$$\frac{e^2-1}{e^2+1} = \frac{1|}{|1} + \frac{1|}{|3} + \frac{1|}{|5} + \frac{1|}{|7} + \cdots + \frac{1\quad|}{|2n+1} + \cdots$$

$$\frac{1}{\sqrt{e}-1} = 1 + \frac{2|}{|3} + \frac{4|}{|5} + \frac{6|}{|7} + \frac{8|}{|9} + \cdots + \frac{2n\quad|}{|2n+1} + \cdots$$

$$\frac{1}{e-1} = \frac{1|}{|1} + \frac{2|}{|2} + \frac{3|}{|3} + \frac{4|}{|4} + \cdots + \frac{n|}{|n} + \cdots$$

Leonhard Euler (1707 - 1783)
De fractionibus continuis dissertatio,
Commentarii Acad. Sci. Petropolitanae,
9 (1737), 1744, p. 98–137 ;
Opera Omnia Ser. I vol. **14**,
Commentationes Analyticae, p. 187–215.

$$
\begin{aligned}
e &= \lim_{n\to\infty} (1+1/n)^n \\
&= 2.718\,281\,828\,459\,045\,235\,360\,287\,471\,352\dots \\
&= 1 + 1 + \frac{1}{2}\cdot\left(1 + \frac{1}{3}\cdot\left(1 + \frac{1}{4}\cdot\left(1 + \frac{1}{5}\cdot(1+\cdots)\right)\right)\right).
\end{aligned}
$$

Continued fraction expansion for e

$$\begin{aligned} e &= [2\,;\ 1,\ 2,\ 1,\ 1,\ 4,\ 1,\ 1,\ 6, \dots] \\ &= [2; \overline{1,\ 2m,\ 1}]_{m \geq 1}. \end{aligned}$$

e is neither rational (J-H. Lambert, 1766) nor quadratic irrational (J-L. Lagrange, 1770).

Continued fraction expansion for $e^{1/a}$

Starting point : $y = \tanh(x/a)$ satisfies the differential equation $ay' + y^2 = 1$.
This leads Euler to

$$e^{1/a} = [1\,;\; a-1,\; 1,\; 1,\; 3a-1,\; 1,\; 1,\; 5a-1, \ldots]$$
$$= [\overline{1,\; (2m+1)a-1,\; 1}]_{m\geq 0}.$$

Geometric proof of the irrationality of e

Jonathan Sondow
http://home.earthlink.net/~jsondow/
A geometric proof that e is irrational and a new measure of its irrationality, Amer. Math. Monthly **113** (2006) 637-641.

Start with an interval I_1 with length 1. The interval I_n will be obtained by splitting the interval I_{n-1} into n intervals of the same length, so that the length of I_n will be $1/n!$.

Geometric proof of the irrationality of e

The origin of I_n will be

$$1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}.$$

Hence we start from the interval $I_1=[2,3]$. For $n\geq 2$, we construct I_n inductively as follows : split I_{n-1} into n intervals of the same length, and call the second one I_n :

$$I_1=\left[1+\frac{1}{1!}\,,\,1+\frac{2}{1!}\right]=[2,3],$$

$$I_2=\left[1+\frac{1}{1!}+\frac{1}{2!}\,,\,1+\frac{1}{1!}+\frac{2}{2!}\right]=\left[\frac{5}{2!}\,,\,\frac{6}{2!}\right],$$

$$I_3=\left[1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}\,,\,1+\frac{1}{1!}+\frac{1}{2!}+\frac{2}{3!}\right]=\left[\frac{16}{3!}\,,\,\frac{17}{3!}\right].$$

The origin of I_n is

$$1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} = \frac{a_n}{n!},$$

the length is $1/n!$, hence $I_n = [a_n/n!, (a_n+1)/n!]$.

The number e is the intersection point of all these intervals, hence it is inside each I_n, therefore it cannot be written $a/n!$ with a an integer.
Since

$$\frac{p}{q} = \frac{(q-1)!\,p}{q!},$$

we deduce that the number e is irrational.

Irrationality measure for e, following J. Sondow

For any integer $n > 1$,

$$\frac{1}{(n+1)!} < \min_{m \in \mathbf{Z}} \left| e - \frac{m}{n!} \right| < \frac{1}{n!}.$$

Smarandache function : $S(q)$ is the least positive integer such that $S(q)!$ is a multiple of q :

$$S(1) = 1,\ S(2) = 2,\ S(3) = 3,\ S(4) = 4,\ S(5) = 5,\ S(6) = 3 \ldots$$

$S(p) = p$ for p prime. Also $S(n!) = n$.
Irrationality measure for e : for $q > 1$,

$$\left| e - \frac{p}{q} \right| > \frac{1}{(S(q)+1)!}.$$

Joseph Fourier

Course of analysis at the École Polytechnique Paris, 1815.

Irrationality of e, following J. Fourier

$$e = \sum_{n=0}^{N} \frac{1}{n!} + \sum_{m \geq N+1} \frac{1}{m!}.$$

Multiply by $N!$ and set

$$B_N = N!, \qquad A_N = \sum_{n=0}^{N} \frac{N!}{n!}, \quad R_N = \sum_{m \geq N+1} \frac{N!}{m!},$$

so that $B_N e = A_N + R_N$. Then A_N and B_N are in $\mathbf{Z}$, $R_N > 0$ and

$$R_N = \frac{1}{N+1} + \frac{1}{(N+1)(N+2)} + \cdots < \frac{e}{N+1}.$$

Irrationality of e, following J. Fourier

In the formula

$$B_N e - A_N = R_N,$$

the numbers A_N and $B_N = N!$ are integers, while the right hand side is > 0 and tends to 0 when N tends to infinity.
Hence $N!\, e$ is not an integer, therefore e is irrational.

Since e is irrational, the same is true for $e^{1/b}$ when b is a positive integer. That e^2 is irrational is a stronger statement.

The number e is not quadratic

Recall (Euler, 1737) : $e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \ldots]$ which is not a periodic expansion. J.L. Lagrange (1770) : it follows that e is not a quadratic number.

Assume $ae^2 + be + c = 0$. Then

$$cN! + \sum_{n=0}^{N} (2^n a + b) \frac{N!}{n!} = -\sum_{k \geq 0} \left(2^{N+1+k} a + b\right) \frac{N!}{(N+1+k)!} \cdot$$

The left hand side is an integer, the right hand side tends to infinity. **It does not work !**

e is not a quadratic irrationality (Liouville, 1840)

Write the quadratic equation as $ae + b + ce^{-1} = 0$.

$$bN! + \sum_{n=0}^{N} (a+(-1)^n c) \frac{N!}{n!}$$
$$= -\sum_{k\geq 0} \left(a + (-1)^{N+1+k} c\right) \frac{N!}{(N+1+k)!}.$$

Using the same argument, we deduce that the LHS and RHS are 0 for any sufficiently large N.

Then ?

e is not quadratic (end of the proof)

Write

$$\sum_{k\geq 0}\left(a+(-1)^{N+1+k}c\right)\frac{N!}{(N+1+k)!}=A+B+C=0$$

with

$$\begin{aligned}
A&=\left(a-(-1)^{N}c\right)\frac{1}{N+1} \qquad (k=0),\\
B&=\left(a+(-1)^{N}c\right)\frac{1}{(N+1)(N+2)} \qquad (k=1),\\
C&=\sum_{k\geq 2}\left(a+(-1)^{N+1+k}c\right)\frac{N!}{(N+1+k)!}.
\end{aligned}$$

For sufficiently large N, we have $A=B=C=0$, hence $a-(-1)^{N}c=0$ and $a+(-1)^{N}c=0$, therefore $a=c=0$ and $b=0$.

The number e^2 is not quadratic

J. Liouville (1809 - 1882) proved that e^2 is not a quadratic irrational number in 1840.

Sur l'irrationalité du nombre $e = 2,718\ldots$, J. Math. Pures Appl. (1) **5** (1840), p. 192 and p. 193-194.

The irrationality of e^4, hence of $e^{4/b}$ for b a positive integer, follows.

e^2 is not quadratic, following Liouville

Write $ae^2 + b + ce^{-2} = 0$ and

$$\frac{N!b}{2^{N-1}} + \sum_{n=0}^{N} (a + (-1)^n c) \frac{N!}{2^{N-n-1} n!} = -\sum_{k \geq 0} \left(a + (-1)^{N+1+k} c \right) \frac{2^k N!}{(N+1+k)!}.$$

It suffices now to check that the numbers

$$\frac{N!}{2^{N-n-1} n!}, \quad (0 \leq n \leq N)$$

are integers for infinitely many N.

Limit of the method

D.W. Masser noticed that the preceding proofs lead to the irrationality of $\theta = e^{\sqrt{2}} + e^{-\sqrt{2}}$, hence of $e^{\sqrt{2}}$.

In the same way we obtain the irrationality of $\sqrt{2}(e^{\sqrt{2}} - e^{-\sqrt{2}})$, but one does not deduce that $e^{\sqrt{2}}$ is not a quadratic number.

One also gets the irrationality of $e^{\sqrt{3}} + e^{-\sqrt{3}}$, hence of $e^{\sqrt{3}}$.

The domain of application of this method is quite limited.

Going further

It does not seem that this kind of argument will suffice to prove the irrationality of e^3, even less to prove that the number e is not a cubic irrational.

Fourier's argument rests on truncating the exponential series, it amounts to approximate e by $a/N!$ where $a \in \mathbf{Z}$. Better rational approximations exist, involving other denominators than $N!$.

The denominator $N!$ arises when one approximates the exponential series of e^z by polynomials $\sum_{n=1}^{N} z^n/n!$.

Irrationality criterion

Let x be a real number. The following conditions are equivalent.

(i) *x is irrational.*

(ii) *For any $\epsilon > 0$, there exists $p/q \in \mathbf{Q}$ such that*

$$0 < \left| x - \frac{p}{q} \right| < \frac{\epsilon}{q} \cdot$$

(iii) *For any real number $Q > 1$, there exists an integer q in the interval $1 \le q < Q$ and there exists an integer p such that*

$$0 < \left| x - \frac{p}{q} \right| < \frac{1}{qQ} \cdot$$

(iv) *There exist infinitely many $p/q \in \mathbf{Q}$ satisfying*

$$0 < \left| x - \frac{p}{q} \right| < \frac{1}{\sqrt{5}q^2} \cdot$$

Irrationality of $\zeta(3)$, following Apéry (1978)

There exist two sequences of rational numbers $(a_n)_{n\geq 0}$ *and* $(b_n)_{n\geq 0}$ *such that* $a_n \in \mathbf{Z}$ *and* $d_n^3 b_n \in \mathbf{Z}$ *for all* $n \geq 0$ *and*

$$\lim_{n\to\infty} |2a_n\zeta(3) - b_n|^{1/n} = (\sqrt{2}-1)^4,$$

where d_n *is the lcm of* $1, 2, \ldots, n$.

We have $d_n = e^{n+o(n)}$ and $e^3(\sqrt{2}-1)^4 < 1$.

References

S. Fischler
Irrationalité de valeurs de zêta, (d'après Apéry, Rivoal, ...),
Sém. Nicolas Bourbaki, 2002-2003,
N° 910 (Novembre 2002).
http://www.math.u-psud.fr/~fischler/publi.html

C. Krattenthaler and T. Rivoal, *Hypergéométrie et fonction zêta de Riemann,* Mem. Amer. Math. Soc. **186** (2007), 93 p.
http://www-fourier.ujf-grenoble.fr/~rivoal/articles.html

Idea of Ch. Hermite

Ch. Hermite (1822 - 1901).
approximate the exponential function e^z by rational fractions $A(z)/B(z)$.

For proving the irrationality of e^a, (a an integer ≥ 2), approximate e^a par $A(a)/B(a)$.

If the function $B(z)e^z - A(z)$ has a zero of high multiplicity at the origin, then this function has a small modulus near 0, hence at $z = a$. Therefore $|B(a)e^a - A(a)|$ is small.

Charles Hermite

A rational function $A(z)/B(z)$ is *close* to a complex analytic function f if both Taylor expansions at the origin match up to a *high* order.
When $f(0) \neq 0$, this means that $B(z)f(z) - A(z)$ has a zero of high multiplicity at the origin.

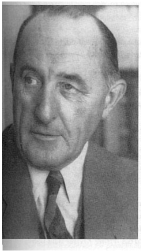
Goal : find $B \in \mathbf{C}[z]$ such that the Taylor expansion at the origin of $B(z)f(z)$ has a big gap : $A(z)$ will be the part of the expansion before the gap, $R(z) = B(z)f(z) - A(z)$ the remainder.

Irrationality of e^r and π *(Lambert, 1766)*

Charles Hermite (1873)

Carl Ludwig Siegel (1929, 1949)

Yuri Nesterenko (2005)



Irrationality of e^r and π *(Lambert, 1766)*

We wish to prove the irrationality of e^a for a a positive integer.

Goal : write $B_n(z)e^z = A_n(z) + R_n(z)$ with A_n and B_n in $\mathbf{Z}[z]$ and $R_n(a) \neq 0$, $\lim_{n\to\infty} R_n(a) = 0$.

Substitute $z = a$, set $q = B_n(a)$, $p = A_n(a)$ and get

$$0 < |qe^a - p| < \epsilon.$$

Rational approximation to exp

Given $n_0 \geq 0$, $n_1 \geq 0$, find A and B in $\mathbf{R}[z]$ of degrees $\leq n_0$ and $\leq n_1$ such that $R(z) = B(z)e^z - A(z)$ has a zero at the origin of multiplicity $\geq N+1$ with $N = n_0 + n_1$.

Theorem *There is a non-trivial solution, it is unique with B monic. Further, B is in $\mathbf{Z}[z]$ and $(n_0!/n_1!)A$ is in $\mathbf{Z}[z]$. Furthermore A has degree n_0, B has degree n_1 and R has multiplicity exactly $N+1$ at the origin.*

$B(z)e^z = A(z) + R(z)$

Proof. Unicity of R, hence of A and B.
Let $D = d/dz$. Since A has degree $\leq n_0$,

$$D^{n_0+1}R = D^{n_0+1}(B(z)e^z)$$

is the product of e^z with a polynomial of the same degree as the degree of B and same leading coefficient.
Since $D^{n_0+1}R(z)$ has a zero of multiplicity $\geq n_1$ at the origin, $D^{n_0+1}R = z^{n_1}e^z$. Hence R is the unique function satisfying $D^{n_0+1}R = z^{n_1}e^z$ with a zero of multiplicity $\geq n_0$ at 0 and B has degree n_1.

Siegel's algebraic point of view

C.L. Siegel, 1949.

Solve $D^{n_0+1}R(z) = z^{n_1}e^z$.

The operator $J\varphi = \displaystyle\int_0^z \varphi(t)dt$,

inverse of D, satisfies

$$J^{n+1}\varphi = \int_0^z \frac{1}{n!}(z-t)^n\varphi(t)dt.$$

Hence

$$R(z) = \frac{1}{n_0!}\int_0^z (z-t)^{n_0}t^{n_1}e^t dt.$$

Also $A(z) = -(-1+D)^{-n_1-1}z^{n_0}$ and
$B(z) = (1+D)^{-n_0-1}z^{n_1}$.

How to get a zero coefficient in the Taylor expansion

$$f(z) = \sum_{k\geq 0} a_k z^k,$$

$$(zf')^{(m)}(0) = \frac{a_m}{(m-1)!}\cdot$$

The coefficient of z^m in the Taylor expansion of $zf'(z) - mf(z)$ is 0.
Writing

$$zf'(z) = \sum_{k\geq 0} ka_k z^k,$$

we have

$$zf'(z) - mf(z) = \sum_{k\geq 0} (k-m)a_k z^k.$$

How to produce several zero coefficients

Set $\delta = zd/dz$, so that $\delta(z^k) = kz^k$. Then $\delta^m z^k = k^m z^k$ for $m \geq 0$. By linearity, if $T \in \mathbf{C}[z]$

$$T(\delta)z^k = T(k)z^k \quad \text{and} \quad T(\delta)f(z) = \sum_{k\geq 0} a_k T(k) z^k.$$

We get a zero coefficient in the Taylor expansion of z^k by considering $T(\delta)f(z)$ where T satisfies $T(k) = 0$. Let $m \geq n$ be integers and

$$T(z) = (z-n-1)(z-n-2)\cdots(z-m).$$

Then $T(\delta)f(z) = A(z) + R(z)$ with

$$A(z) = \sum_{k=0}^{n} T(k)a_k z^k \quad \text{and} \quad R(z) = \sum_{k\geq m+1} T(k)a_k z^k.$$

In the Taylor expansion of $T(\delta)f(z)$, the coefficients of $z^{n+1}, z^{n+2}, \ldots, z^m$ are 0.

Take $f(z) = e^z$ with $a_k = 1/k!$ and $m = 2n$. Let

$$T_n(z) = (z-n-1)(z-n-2)\cdots(z-2n).$$

We have

$$\delta(e^z) = ze^z.$$

There exists $B_n \in \mathbf{Z}[z]$, monic of degree n, such that $T_n(\delta)e^z = B_n(z)e^z$. Hence

$$\boxed{B_n(z)e^z = A_n(z) + R_n(z)}$$

with

$$A_n(z) = \sum_{k=0}^{n} T_n(k)\frac{z^k}{k!} \quad \text{and} \quad R_n(z) = \sum_{k\geq 2n+1} T_n(k)\frac{z^k}{k!}.$$

The coefficients of A_n are integers

Each coefficient of A_n is a multiple of a binomial coefficient

$$\frac{T_n(k)}{k!} = (-1)^n(2n-k)(2n-k-1)\cdots(n+1) \cdot \frac{n(n-1)\cdots(n-k+1)}{k!}$$

for $0 \leq k \leq n$. Hence $A_n \in \mathbf{Z}[z]$.

On the other hand one checks

$$\lim_{n\to\infty} R_n(z) = 0.$$

Irrationality of e^r, following Yu. V.Nesterenko

Let a be a positive integer. Set $s = e^a$. Replacing z by a, we get

$$B_n(a)s - A_n(a) = R_n(a).$$

The coefficients of R_n are all positive, hence $R_n(a) > 0$ and $B_n(a)s - A_n(a) \neq 0$. Since $R_n(a)$ tends to 0 when n tends to infinity, and since $B_n(a)$ and $A_n(a)$ are integers, one deduces that s is irrational.

Irrationality of logarithms including π

The irrationality of e^r for $r \in \mathbf{Q}^\times$, is equivalent to the irrationality of $\log s$ for $s \in \mathbf{Q}_{>0}$.

The same argument gives the irrationality of $\log(-1)$, meaning $\log(-1) = i\pi \notin \mathbf{Q}(i)$.

Irrationality of π, following Yu. V.Nesterenko

Assume π is a rational number, $\pi = a/b$. Replace z by $ia = i\pi b$ in the previous formulae. Since $e^z = (-1)^b$ we have

$$B_n(ia)(-1)^b - A_n(ia) = R_n(ia),$$

and the two complex numbers $A_n(ia)$ and $B_n(ia)$ are in $\mathbf{Z}[i]$. The left hand side is in $\mathbf{Z}[i]$, the right hand side tends to 0 when n tends to infinity Hence both vanish

Using a resultant, one shows that R_n and R_{n+1} have no common zero apart from 0. Hence the contradiction.

Irrationality of π^2

A short proof of the irrationality of π^2 (J. Niven 1946)
Assume $\pi^2 = p/q$. For a sufficiently large positive integer n, consider

$$a_n = \pi \cdot \frac{p^n}{n!} \int_0^1 \sin(\pi x) x^n (1-x)^n dx.$$

Then $a_n \in \mathbf{Z}$ et $0 < a_n < 1$. Contradiction.

In the same way, for the irrationality of e^k, $k \in \mathbf{Z}$, $k \neq 0$ (hence the irrationality of e^r for $r \in \mathbf{Q}$, $r \neq 0$) : if $e^k = p/q$, consider

$$qk^{2n+1} \int_0^1 e^{kx} x^n (1-x)^n dx.$$

1979, F. Beukers : $\zeta(2) \notin \mathbf{Q}$ (R. Apéry, 1978, $\zeta(3) \notin \mathbf{Q}$).

Historical introduction to irrationality

Michel Waldschmidt

http://www.math.jussieu.fr/∼miw/